18.745 Problem Set 4

arr. Swapnil Garg

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1 Lecture 8

1. Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{F}), n \geq 2$, char $\mathbb{F} \neq 2$. Let $\mathfrak{h} = \mathbb{F}I_n + \mathfrak{n}_n$. Then $\mathfrak{h}$ is a maximal nilpotent subalgebra but not a Cartan subalgebra.
2. All nilpotent Lie algebras of dimension 3 are the abelian Lie algebra and the Heisenberg Lie algebra.
3. Let $\dim \mathfrak{g} = 3, \dim \mathfrak{h} = 1, \mathfrak{h}$ a Cartan subalgebra. Show that $\mathfrak{g}$ is isomorphic to one of the 3 cases (in each case $\mathbb{F}h$ is a Cartan subalgebra)
   (a) $[h, a] = a, [h, b] = a + b, [a, b] = 0$
   (b) $[h, a] = a, [h, b] = \lambda b, [a, b] = 0, \lambda \neq 0$
   (c) $[h, a] = a, [h, b] = -b, [a, b] = h$.
4. Show that the three cases from Exercise 8.3 are nonisomorphic.

2 Lecture 9

1. if A, B are commuting nilpotent operators, then $e^{A+B} = e^A e^B$. In particular, $e^A e^{-A} = I$, so e^A is nonsingular.
2. Let D be a derivation of an algebra $\mathfrak{g}$ (not necessarily Lie), which is a nilpotent operator. Prove that e^D is an automorphism of $\mathfrak{g}$.
3. Chevalley's Lemma is the following: Let $f : \mathbb{F}^m \Rightarrow \mathbb{F}^m$ be a polynomial map with $\mathbb{F}$ algebraically closed. Suppose that the linear map $(df)|_{x=a} : \mathbb{F}^m \Rightarrow \mathbb{F}^m$ is nonsingular, for some a. Then $f(\mathbb{F}^m)$ contain s a nonempty Zariski open subset of $\mathbb{F}^n$. Prove Chevalley's lemma by the following steps.
   (a) $(df)_{x=a}$ is a linear map $\mathbb{F}^m \Rightarrow \mathbb{F}^m$, given by the Jacobian matrix $(\frac{\partial f_i}{\partial x_j} a)_{i,j=1}^m$.
   (b) If $F(f_1, \ldots, f_m) \equiv 0$ for some nonzero polynomial F in m variables, then det $(\frac{\partial f_i}{\partial x_j})_{i,j=1}^m \equiv 0$.
   (c) Given algebraically independent elements $y_1, \ldots, y_m \in \mathbb{F}[x_1, \ldots, x_m]$, show that the field extension $\mathbb{F}(y_1, \ldots, y_m) \subset \mathbb{F}(x_1, \ldots, x_m)$ is finite, i.e. each x_i satisfies a nonzero polynomial over the field $\mathbb{F}(y_1, \ldots, y_m)$.
   (d) For each $i = 1..m$, pick a polynomial equation satisfied by x_i over $\mathbb{F}(f_1, \ldots, f_m)$, clear the denominators to get a polynomial, and let $p_i(f_1, \ldots, f_m)$ be the leading coefficient of this polynomial. Then $\mathbb{F}^m \backslash f \cup (p_1, \ldots, p_m)$ is a nonempty Zariski open set claimed by Chevalley's Lemma.

3 Lecture 10

1. Let $\mathbb{F}$ be a field of characteristic 0, D a nilpotent derivation. Then e^D is an automorphism of $\mathfrak{g}$. Show that if (a, b) is an invariant bilinear form on $\mathfrak{g}$, then $(e^D a, e^D b) = (a, b)$.

2. Show that the trace form on $\mathfrak{gl}_n, \mathfrak{sl}_n$ in the standard n-dimensional representation is nondegenerate. The Killing form on $\mathfrak{sl}_n$ is nondegenerate, provided that char $\mathbb{F}$ does not divide $2N$. Find the radical of the Killing form on $\mathfrak{gl}_n$.

3. Cartan's criterion states the following: Let $\mathfrak{g}$ be a subalgebra of $\mathfrak{gl}_V$ for V finite dimensional over an algebraically closed field of characteristic 0. Then the following are equivalent:

   (a) $(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}])_v = 0$

   (b) $(a, a)_V = 0$ for all a in $[\mathfrak{g}, \mathfrak{g}]$

   (c) $\mathfrak{g}$ is solvable

   The corollay states that a finite dimensional Lie algebra over an algebraically closed field with characteristic 0 is solvable iff $\kappa(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$.

   Consider the diamond Lie algebra : $D = Heis_3 + \mathbb{F}d$, with $[p, q] = c, [d, p] = p, [d, q] = -q$, c central. This is a solvable Lie algebra. Define a symmetric bilinear form on D by $(p, q) = 1, (c, d) = 1$, and everything else is 0. Show that this form is invariant but it does not satisfy Cartan's criterion, so it is not a trace form.

4. Let $\bar{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{F}} \bar{\mathbb{F}}$.

   (a) Prove that $\mathfrak{g}$ is solvable (resp. nilpotent) iff $\bar{\mathfrak{g}}$ is solvable (resp. nilpotent.

   (b) Derive Cartan's criterion and corollary for arbitrary $\mathbb{F}$ of characteristic 0.

   (c) Derive that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent if $\mathfrak{g}$ is solvable for $\mathbb{F}$ of characteristic 0.

   (d) Derive that $\mathfrak{g}_0^a$ is a Cartan subalgebra for any regular element a for $\mathbb{F}$ of characteristic 0.