1 Lecture 8

1. Let $g = \mathfrak{g}_n(F), n \geq 2, \text{char } F \neq 2$. Let $\mathfrak{h} = FI_n + n_n$. Then $\mathfrak{h}$ is a maximal nilpotent subalgebra but not a Cartan subalgebra.

2. All nilpotent Lie algebras of dimension 3 are the abelian Lie algebra and the Heisenberg Lie algebra.

3. Let $\dim g = 3, \dim h = 1, h$ a Cartan subalgebra. Show that $g$ is isomorphic to one of the 3 cases (in each case $Fh$ is a Cartan subalgebra)

   (a) $[h, a] = a, [h, b] = a + b, [a, b] = 0$
   (b) $[h, a] = a, [h, b] = \lambda b, [a, b] = 0, \lambda \neq 0$
   (c) $[h, a] = a, [h, b] = -b, [a, b] = h$.

4. Show that the three cases from Exercise 8.3 are nonisomorphic.

2 Lecture 9

1. If $A, B$ are commuting nilpotent operators, then $e^{A+B} = e^Ae^B$. In particular, $e^Ae^{-A} = I$, so $e^A$ is nonsingular.

2. Let $D$ be a derivation of an algebra $g$ (not necessarily Lie), which is a nilpotent operator. Prove that $e^D$ is an automorphism of $g$.

3. Chevalley’s Lemma is the following: Let $f : F^m \to F^m$ be a polynomial map with $F$ algebraically closed. Suppose that the linear map $(df)|_{x=a} : F^m \to F^m$ is nonsingular, for some $a$. Then $f(F^m)$ contains a nonempty Zariski open subset of $F^m$. Prove Chevalley’s lemma by the following steps.

   (a) $(df)|_{x=a}$ is a linear map $F^m \to F^m$, given by the Jacobian matrix $(\frac{\partial f_i}{\partial x_j}a)_{i,j=1}^m$.
   (b) If $F(f_1, \ldots, f_m) \equiv 0$ for some nonzero polynomial $F$ in $m$ variables, then $\det (\frac{\partial f_i}{\partial x_j})_{i,j=1}^m \equiv 0$.
   (c) Given algebraically independent elements $y_1, \ldots, y_m \in \mathbb{F}[x_1, \ldots, x_m]$, show that the field extension $\mathbb{F}(y_1, \ldots, y_m) \subset \mathbb{F}(x_1, \ldots, x_m)$ is finite, i.e. each $x_i$ satisfies a nonzero polynomial over the field $\mathbb{F}(y_1, \ldots, y_m)$.
   (d) For each $i = 1, m$, pick a polynomial equation satisfied by $x_i$ over $\mathbb{F}(f_1, \ldots, f_m)$, clear the denominators to get a polynomial, and let $p_i(f_1, \ldots, f_m)$ be the leading coefficient of this polynomial. Then $\mathbb{F}^m \setminus f \cup (p_1, \ldots, p_m)$ is a nonempty Zariski open set claimed by Chevalley’s Lemma.

3 Lecture 10

1. Let $\mathbb{F}$ be a field of characteristic 0, $D$ a nilpotent derivation. Then $e^D$ is an automorphism of $g$. Show that if $(a, b)$ is an invariant bilinear form on $g$, then $(e^D a, e^D b) = (a, b)$. 
2. Show that the trace form on $\text{gl}_n, \text{sl}_n$ in the standard $n$-dimensional representation is nondegenerate. The Killing form on $\text{sl}_n$ is nondegenerate, provided that char $F$ does not divide $2N$. Find the radical of the Killing form on $\text{gl}_n$.

3. Cartan’s criterion states the following: Let $g$ be a subalgebra of $\text{gl}_V$ for $V$ finite dimensional over an algebraically closed field of characteristic 0. Then the following are equivalent:

(a) $(g, [g, g])_v = 0$
(b) $(a, a)_v = 0$ for all $a$ in $[g, g]$
(c) $g$ is solvable

The corollary states that a finite dimensional Lie algebra over an algebraically closed field with characteristic 0 is solvable iff $\kappa(g, [g, g]) = 0$.

Consider the diamond Lie algebra: $D = \text{Heis}_3 + \mathbb{F}d$, with $[p, q] = c, [d, p] = p, [d, q] = -q, c$ central. This is a solvable Lie algebra. Define a symmetric bilinear form on $D$ by $(p, q) = 1, (c, d) = 1$, and everything else is 0. Show that this form is invariant but it does not satisfy Cartan’s criterion, so it is not a trace form.

4. Let $\bar{g} = g \otimes_F \bar{F}$.

(a) Prove that $g$ is solvable (resp. nilpotent) iff $\bar{g}$ is solvable (resp. nilpotent).
(b) Derive Cartan’s criterion and corollary for arbitrary $F$ of characteristic 0.
(c) Derive that $[g, g]$ is nilpotent if $g$ is solvable for $F$ of characteristic 0.
(d) Derive that $g^0_\alpha$ is a Cartan subalgebra for any regular element $\alpha$ for $F$ of characteristic 0.