18.745 Problem Set 3

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1 Lecture 6

1. Let char $\mathbb{F} = 2$, $V = \mathbb{F}[x]/(x^2)$ be a representation of Heis_3 defined by $p \mapsto \frac{d}{dx}$, $q \mapsto$ multiply by x, $c \mapsto 1$. Then, $V = V_\lambda$ but λ is not linear on Heis_3. Compute the function λ. Note that for an operator A, V_λ is the set of vectors v such that $(A - \lambda I)^N(v) = 0$ for some positive integer N.

2. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, π a representation in a finite dimensional vector space V over an algebraically closed field of characteristic 0, and $\mathfrak{h}$ a nilpotent subalgebra of $\mathfrak{g}$.
   Let $V_\lambda^{\mathfrak{h}}$ be the set of vectors in V such that for any element h of $\mathfrak{h}$, $(\pi(h) - \lambda(h)I)^N(v) = 0$ for some positive integer N.

   Similarly, $\mathfrak{g}_\alpha^{\mathfrak{h}}$ is the set of elements in $\mathfrak{g}$ such that for any element g in $\mathfrak{g}$, $(\text{ad}\, a - \alpha(a)I)^N g = 0$ for all a in $\mathfrak{h}$ and a positive integer N.

   Then we have a decomposition (standard weight space decomposition) $V = \oplus V_\lambda^{\mathfrak{h}}$ and $g = \oplus \alpha^{\mathfrak{h}}$, with $\pi(g_\alpha^{\mathfrak{h}})V_\lambda^{\mathfrak{h}} \subset V_{\lambda+\alpha}^{\mathfrak{h}}$.
   By the example of the adjoint representation of a 2-dimensional nonabelian Lie algebra, show that the above theorem fails for solvable Lie algebras (instead of the stronger nilpotent condition).

3. Find the generalized weight space decomposition for the standard representation of $\mathfrak{gl}_N$ on $\mathbb{F}^N$ with respect to $\mathfrak{h}$ = diagonal matrices. Do the same for the adjoint representation.

2 Lecture 7

1. Let $X = \mathbb{F}^n$. The Zariski topology is defined as the closed sets being the sets of common zeros in a (possibly infinite) collection of polynomials in $x_1, x_2, ..., x_n$. Prove that this is a topology.

2. Let $\mathfrak{g}$ be a d-dimensional Lie algebra over $\mathbb{F}$. Consider the characteristic polynomial of $\text{ad}\, a$, for $a \in \mathfrak{g}$:

$$\det(\text{ad}\, a - \lambda) = (-\lambda)^d + c_{d-1}(a)(-\lambda)^{d-1} + ... + \det(\text{ad}\, a).$$

   Show that $c_j(a)$ is a homogeneous polynomial on $\mathfrak{g}$ of degree $d - j$, i.e. if we wrote $a = \sum_{i=1}^d x_i e_i$ for $\{e_i\}$ a basis of $\mathfrak{g}$, then c_j are homogeneous polynomials in $x_1, x_2, ..., x_n$ of degree $d - j$.

3. Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{F})$, where $\mathbb{F}$ is algebraically closed. For $a \in g$, let $a_s + a_n$ be its Jordan decomposition. Prove the following:

   (a) $\text{ad}\, a = \text{ad}\, a_s + \text{ad}\, a_n$ is Jordan decomposition.

   (b) If $\lambda_1, ..., \lambda_n$ are the eigenvalues of a (and a_s), then all $\lambda_i - \lambda_j$ are the eigenvalues of $\text{ad}\, a$ (and $\text{ad}\, a_s$).

4. The rank of $\mathfrak{g}$ is the smallest r such that $c_r(a)$ is a nonzero polynomial. Deduce from 7.3 that the rank of $\mathfrak{g}$ is n and that the discriminant $c_n(a) = \prod_{i \neq j}(\lambda_i - \lambda_j)$.