18.745 Problem Set 2

arr. Swapnil Garg

September 2018

1 Lecture 4

1. Let n_N be the subspace of $\mathfrak{gl}_N(\mathbb{F})$ of strictly upper triangular matrices, and let b_N be the subspace of non-strictly upper triangular matrices. Show that b_N is solvable, that $[b_N, b_N] = n_N$, and that n_N is nilpotent.

2. Let $\mathfrak{h}$ be an ideal of g, where g is a Lie algebra. Show that if $\mathfrak{h}$ and $g/\mathfrak{h}$ are solvable, then g is solvable.

3. Show that if g is nonabelian 2-step nilpotent, finite-dimensional Lie algebra, and the dimension of $Z(g)$ is 1, then g is isomorphic to a Heisenberg Lie algebra.

2 Lecture 5

1. Show that any subspace of a Lie algebra g containing $[g, g]$ is an ideal of g.

2. Lie's Lemma states the following: If g is a Lie algebra, and $\mathfrak{h} \subset g$ an ideal of g, both over a field $\mathbb{F}$ with characteristic 0. Let $\pi : g \Rightarrow \mathfrak{gl}_V$ be a representation of g in a finite dimensional vector space V over $\mathbb{F}$. Then, each weight space $V_\lambda^{\mathfrak{h}}$ for π restricted to $\mathfrak{h}$ is invariant under g: $\pi(g)V_\lambda^{\mathfrak{h}} \subset V_\lambda^{\mathfrak{h}}$.

   Show that Lie's Lemma holds for $\mathbb{F}$ having characteristic p, provided that $\dim V < p$.

3. A counterexample to Lie's theorem if char $\mathbb{F} = p$: Recall that Heis$_3$ has a representation π in $\mathbb{F}[x]$ with $\pi(c) = 1, \pi(p) = x, \pi(q) = \frac{d}{dx}$. Show that π is a representation, and that $J = span\{x^p, x^{p+1}, \ldots\}$ is a subrepresentation. So, we have a representation of Heis$_3$ in $\mathbb{F}[x]/J$ of dimension p. Show that this representation has no weight, i.e. $\pi(p), \pi(q)$ have no common eigenvector.

4. Prove the following corollaries of Lie's Theorem.
   a. For all representations π of a solvable lie algebra g over an algebraically

closed field $\mathbb{F}$ with characteristic 0, there exists a basis of V in which all $\pi(a)$ for $a \in g$ are nonstrictly upper triangular.

b. Any solvable subalgebra $g \in \mathfrak{gl}_V$, for finite dimensional V over an algebraically closed field $\mathbb{F}$ with characteristic 0, is contained in the subalgebra b_N for some basis of V.

5. If g is a finite dimensional, solvable Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0, then $[g, g]$ is nilpotent.