1 Lecture 4

1. Let $n_N$ be the subspace of $\mathfrak{gl}_N(\mathbb{F})$ of strictly upper triangular matrices, and let $b_N$ be the subspace of non-strictly upper triangular matrices. Show that $b_N$ is solvable, that $[b_N,b_N] = n_N$, and that $n_N$ is nilpotent.

2. Let $\mathfrak{h}$ be an ideal of $g$, where $g$ is a Lie algebra. Show that if $\mathfrak{h}$ and $g/\mathfrak{h}$ are solvable, then $g$ is solvable.

3. Show that if $g$ is nonabelian 2-step nilpotent, finite-dimensional Lie algebra, and the dimension of $Z(g)$ is 1, then $g$ is isomorphic to a Heisenberg Lie algebra.

2 Lecture 5

1. Show that any subspace of a Lie algebra $g$ containing $[g,g]$ is an ideal of $g$.

2. Lie’s Lemma states the following: If $g$ is a Lie algebra, and $\mathfrak{h} \subset g$ an ideal of $g$, both over a field $\mathbb{F}$ with characteristic 0. Let $\pi : g \rightarrow \mathfrak{gl}_V$ be a representation of $g$ in a finite dimensional vector space $V$ over $\mathbb{F}$. Then, each weight space $V^\lambda$ for $\pi$ restricted to $\mathfrak{h}$ is invariant under $g$: $\pi(g)V^\lambda \subset V^\lambda$.

Show that Lie’s Lemma holds for $\mathbb{F}$ having characteristic $p$, provided that $\dim V < p$.

3. A counterexample to Lie’s theorem if char $\mathbb{F} = p$: Recall that Heis$_3$ has a representation $\pi$ in $\mathbb{F}[x]$ with $\pi(c) = 1, \pi(p) = x, \pi(q) = \frac{d}{dx}$. Show that $\pi$ is a representation, and that $J = \text{span}\{xp, xp+1, \ldots\}$ is a subrepresentation. So, we have a representation of Heis$_3$ in $\mathbb{F}[x]/J$ of dimension $p$. Show that this representation has no weight, i.e. $\pi(p), \pi(q)$ have no common eigenvector.

4. Prove the following corollaries of Lie’s Theorem.
   a. For all representations $\pi$ of a solvable lie algebra $g$ over an algebraically
closed field $\mathbb{F}$ with characteristic 0, there exists a basis of $V$ in which all $\pi(a)$ for $a \in g$ are nonstrictly upper triangular.

b. Any solvable subalgebra $g \in \mathfrak{gl}_V$, for finite dimensional $V$ over an algebraically closed field $\mathbb{F}$ with characteristic 0, is contained in the subalgebra $b_N$ for some basis of $V$.

5. If $g$ is a finite dimensional, solvable Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0, then $[g, g]$ is nilpotent.