18.745 Problem Set 10

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1 Lecture 21

1. Let r_α be the reflection defined by $r_\alpha(\alpha) = -\alpha, r_\alpha(v) = v$ if $(v, \alpha) = 0$.
   (a) $r_\alpha \in O_V(\mathbb{R})$, i.e. $(r_\alpha(u), r_\alpha(v)) = (u, v)$
   (b) $r_{-\alpha} = r_\alpha$, $r_\alpha^2 = 1$
   (c) $\det_V(r_\alpha) = -1$
   (d) If $A \in O_V(\mathbb{R})$, then $Ar_\alpha A^{-1} = r_A(\alpha)$

2. The Weyl group is the group generated by all r_α for α in a root system. Compute the Weyl groups for B_r, C_r, D_r, and show that they are isomorphic for B_r and C_r but not D_r.

3. Consider the open set $V \setminus \cup_{\alpha\in\Delta} T_\alpha, T_\alpha = \{v \in V|(\alpha, v) = 0\}$. The connected components of this set are called open chambers. The connected component of all v with $(\alpha_i, v) = 0$ for all simple roots α_i is called the fundamental chamber. Prove that the open fundamental chamber is an open chamber.

2 Lecture 22

1. For a Lie algebra $\mathfrak{g}$, an enveloping algebra is a pair (U, ϕ) such that U is a unital associative algebra and there is a Lie algebra homomorphism $\phi : \mathfrak{g} \to U_-$, the Lie algebra with bracket $[a, b] = ab - ba$ for $a, b \in U$.
   Prove that there is a unique universal enveloping algebra of $\mathfrak{g}$, (U, Φ), a universal algebra such that for any other enveloping algebra (U, ϕ), then there exists $f : U(\mathfrak{g}) \to U$ such that $f \circ \Phi = \phi$ as Lie algebra homomorphisms.

2. Let $T(\mathfrak{g})$ be the free unital associative algebra on a basis $a_1, a_2, ...$ of $\mathfrak{g}$. Let $J(\mathfrak{g})$ be the 2-sided ideal generated by elements of the form $a_ia_j - a_ja_i - [a_i, a_j]$. Then $U(\mathfrak{g}) = T(\mathfrak{g})/J(\mathfrak{g})$. Prove the universality property for $(U(\mathfrak{g}), \Phi)$ for Φ the canonical map.

3. The Casimir element Ω is the sum of a_ib_i in $U(\mathfrak{g})$, where $\mathfrak{g}$ is semisimple and the b_i are such that for the bilinear form on $\mathfrak{g}$, $(a_i, b_j) = \delta_{ij}$. Prove that Ω is independent of the choice of $\{a_i\}$.

3 Lecture 23

1. Given a $\mathfrak{g}$-module V (basically a representation of $\mathfrak{g}$ where we omit the π, so $\pi(g)v = gv$), a 1-cocycle is a linear map $f : \mathfrak{g} \to V$ such that $f([a, b]) = af(b) - bf(a)$. Prove that the trivial cocycle $f_v(a) = av$ is a 1-cocycle.

2. For a Lie algebra $\mathfrak{g}$ with a nondegenerate invariant bilinear form and dual bases $\{a_i\}, \{b_i\}$, and f a 1-cocyle, Ω the Casimir element, prove that for any $a \in \mathfrak{g}$ we have $a \sum_{j=1}^{\dim \mathfrak{g}} a_jf(b_j) = \Omega f(a)$, using the dual bases lemma from Lecture 22.

3. If $Z^1(\mathfrak{g}, V)$ is the space of all cocycles, and $B^1(\mathfrak{g}, V)$ the space of all trivial cocycles, their quotient is $H^1(\mathfrak{g}, V)$, the first cohomology of $\mathfrak{g}$ with coefficients in V. Prove that $H^1(\mathfrak{g}, V_1 \oplus V_2) = H^1(\mathfrak{g}, V_1) \oplus H^1(\mathfrak{g}, V_2)$.

4. If char $\mathbb{F} = 0$, using the results from the algebraically closed case, prove that $V = V_0 \oplus V'$ where V_0 is the generalized 0-eigenspace of Ω and V' is Ω-invariant.

5. Consider the $\mathfrak{g}$-module End V such that $a \cdot A = aA - Aa$ for $A \in$ End V. Let $M \subset$ End V be a subspace consisting of $A \in$ End V such that $AV \subset U, AU \subset 0$, for a given $U \subset V$. Let P_0 be a projector of V onto U so that $P_0(V) \subset U$, $P_0(U) = 0$. Then M is a submodule of the $\mathfrak{g}$-module End V, and moreover, the cocycle $f = f_{P_0}$ defined by $f_{P_0}(a) = aP_0 - P_0 a$ is actually a cocycle of the $\mathfrak{g}$-module M, i.e. $f(\mathfrak{g}) \subset M$.

4 Lecture 24

1. Recall the triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ for $\mathfrak{g}$ a semisimple Lie algebra over an algebraically closed field of characteristic 0. Then $\mathfrak{b} = \mathfrak{h} \ltimes \mathfrak{n}_+$ is called a Borel subalgebra of $\mathfrak{g}$. Show that it is a maximal solvable subalgebra of $\mathfrak{g}$.

2. Show that if $\{H_i\}, \{H^i\}$ are dual bases of $\mathfrak{h}$, then $\sum_i \lambda(H_i)\lambda(H^i) = (\lambda, \lambda)$.

3. A Verma module $M(\Lambda)$ is a highest weight module with highest weight Λ, which is universal in the sense that any other highest weight module with highest weight Λ is a quotient of $M(\Lambda)$. Prove uniqueness and existence, namely $M(\Lambda) = U(\mathfrak{g})/U(\mathfrak{g})\{\mathfrak{n}_+; h - \Lambda(h), h \in \mathfrak{h}\}$. Then v_Λ is the image of 1 under the map $U(\mathfrak{g}) \to M(\Lambda)$.

   Note: a highest weight module has the property that there exists v_Λ with

   (a) $hv_\Lambda = \Lambda(h)v_\Lambda$ for $h \in \mathfrak{h}$
   (b) $\mathfrak{n}_+ v_\Lambda = 0$
   (c) $U(\mathfrak{g})v_\Lambda = V$

4. Using the PBW Theorem, prove that $E_{-\beta_1}^{m_1} ... E_{-\beta_N}^{m_N} v_\Lambda$ for nonnegative integer m_i form a basis of $M(\Lambda)$.

5 Lecture 25

1. Show that for $\mathfrak{g} = \mathfrak{sl}_2$, an irreducible module $L(\Lambda)$ for $\Lambda(H) = m$ is homogeneous polynomials in x and y of degree m, where $E = x\frac{\partial}{\partial y}, F = y\frac{\partial}{\partial x}, H = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ and the highest weight vector is x^m.

2. For $\mathfrak{g} = \mathfrak{sl}_{r+1}$, show that letting $\Lambda(H_1) = 1, \Lambda(H_i) = 0$ for $i \neq 1$ corresponds to $L(\Lambda)$ being the standard representation, and $\Lambda(H_1) = \lambda(H_r) = 1$, $\Lambda(H_i) = 0$ for $i \neq 1, r$ corresponds $L(\Lambda)$ being the adjoint representation.