18.745 Problem Set 1

arr. Swapnil Garg

September 2018

1 Lecture 1

1. For A and algebra with product $a * b = ab$, denote A_- the algebra with product $[a, b] = ab - ba$. Show A_- is a Lie algebra if any of the following is true, where the statements below have to hold for all $a, b \in A_-$.
   (a) (2-member identity) $(ab)c = a(bc)$
   (b) (3-member identity) $a(bc) + b(ca) + c(ab) = 0$ and $(ab)c + (bc)a + (ca)b = 0$
   (c) (4-member identity) $a(bc) - (ab)c$ is unchanged if we permute a and b (left-symmetric algebra)
   (d) (alternate 4-member identity) $a(bc) - (ab)c$ is unchanged if we permute b and c (right-symmetric algebra)
   (e) (6-member identity) $[a, bc] + [b, ca] + [c, ab] = 0$

2. Let B be a bilinear form on V. Then

$$\mathfrak{o}_{V,B} = \{a \in \mathfrak{gl}_V | B(a(u), v) + B(u, a(v)) = 0 \forall u, v \in V\}.$$

   (a) Show that for $a, b \in \mathfrak{gl}_n(\mathbb{F})$, $\text{tr}[a, b] = 0$.
   (b) Show that $\mathfrak{o}_{V,B}$ is a subalgebra of $\mathfrak{gl}_V$.

3. Show that

$$\mathfrak{o}_{\mathbb{F}_n,B} = \{a | a^T B + Ba = 0\}.$$

4. If $f : \mathfrak{gl}_n(\mathbb{F}) \Rightarrow \mathbb{F}$ is a linear function such that $f([a, b]) = 0$ for any a, b, then $f = \lambda \cdot \text{tr}$ for some $\lambda \in \mathbb{F}$.

2 Lecture 2

1. A *derivation* of an arbitrary algebra is a vector space endomorphism D of A such that $D(ab) = D(a)b + aD(b)$. (This is called the **Leibniz rule.**) Let $\text{Der}(A)$ be the subspace in End A of all derivations of A. Prove that $\text{Der}(A)$ is a subalgebra of gl_A with the usual bracket.

2. Let $A = \mathbb{F}[x_1, x_2, \cdots, x_n]$. Define a bracket on A

$$\{f, g\} = \sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \{x_i, x_j\}$$

   for some choice of $\{x_i, x_j\}$ in A. Prove that this is a Poisson bracket (i.e. the Lie algebra axioms hold) if and only if the skew symmetry axiom holds: $\{x_i, x_i\} = 0$, $\{x_i, x_j\} = -\{x_j, x_i\}$, and the Jacobi identity holds for x_i, x_j, x_k.

3. Let $\phi : g_1 \to g_2$ be a homomorphism. Then show that
   (a) ker ϕ is an ideal of g_1,

(b) im ϕ is a subalgebra of g_2,

(c) im $\phi \simeq g_1/\ker\phi$.

4. An algebraic group G over a field $\mathbb{F}$ is a collection of polynomials $\{P_\alpha\}, \alpha \in I$, on the space of matrices $\mathrm{Mat}_{n\times n}(\mathbb{F})$, such that for any unital commutative associative algebra A over $\mathbb{F}$, the set

$$G(A) := \{g \in \mathrm{Mat}_{n\times n}(A) \mid g \text{ non-singular}, P_\alpha(g) = 0 \forall a \in I\}$$

is a group under the matrix multiplication.

Let $B \in \mathrm{Mat}_{n\times n}(\mathbb{F})$, and let $O_{n,B}(A) = \{g \in GL_n(A) \mid g^T B g = B\}$. Show that this is an algebraic group.

5. The algebra of dual numbers is $D = \mathbb{F}[\epsilon]/(\epsilon^2) = \{a + b\epsilon | \epsilon^2 = 0, a, b \in \mathbb{F}\}$. The Lie algebra Lie G of an algebraic group G is Lie $G = \{X \in \mathfrak{gl}_n(\mathbb{F}) | I_n + \epsilon X \in G(D)\}$.

(a) $(I_n + \epsilon X)^{-1} = I_n - \epsilon X$.

(b) Lie $GL_n = \mathfrak{gl}_n(\mathbb{F})$, Lie $SL_n = \mathfrak{sl}_n(\mathbb{F})$, Lie $O_{n,B} = \mathfrak{o}_{\mathbb{F}^n,B}$.

3 Lecture 3

1. The center of a lie algebra is $Z(g) = \{c \in g | [c, a] = 0 \forall a \in g\}$. Prove that $Z(\mathfrak{gl}_n(\mathbb{F}) = \mathbb{F}I_n, Z(\mathfrak{sl}_n(\mathbb{F}) = \mathbb{F}$ if n does not divide char $\mathbb{F}$, and 0 otherwise.

2. For a finite dimensional Lie algebra g, dim $Z(g) \neq$ dim $g - 1$.

3. dim $Z(g) =$ dim $g - 2$ in exactly the following cases:

(a) $g = b \oplus Ab_{n-2}$ where b is a two-dimensional non-abelian Lie algebra, and Ab_m is an abelian Lie algebra with m dimensions.

(b) $g = Heis_3 \oplus Ab_{n-3}$, where $Heis_{2n+1}$ is the Lie algebra with basis p_i, q_i, c for $1 \leq i \leq n$ and brackets $[p_i, q_i] = -[q_i, p_i] = c$, with all other brackets 0.

4. For finite dimensional V, show that $A \in$ End V is nilpotent iff all eigenvalues are 0.

5. Engel's Theorem states that if $g \subset \mathfrak{gl}_V$ is a finite-dimensional subalgebra consisting only of nilpotent isomorphisms (but not necessarily all of them) and V is nonzero, then there exists $v \neq 0$ in V that is killed by all endomorphisms in g. Deduce from Engel's Theorem that if $\pi : g \Rightarrow \mathfrak{gl}_V$ is a Lie algebra representation of g in V, for a finite dimensional V, then there exists a basis of V in which all operators $\pi(a), a \in g$ have strictly upper triangulator matrices. Hint: dim $\pi(g) \leq$ dim End $V \leq$ (dim $V)^2$.

6. It is important in Engel's Theorem that g is a subalgebra, not just a subspace. Show that $\mathbb{F} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $+ \mathbb{F} \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ consists of nilpotent matrices, but there is no common eigenvector.