18.745 Problem Set 1
arr. Swapnil Garg
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1 Lecture 1

1. For $A$ and algebra with product $a \ast b = ab$, denote $A_-$ the algebra with product $[a, b] = ab - ba$. Show $A_-$ is a Lie algebra if any of the following is true, where the statements below have to hold for all $a, b \in A$.
   
   (a) (2-member identity) $(ab)c = a(bc)$
   (b) (3-member identity) $a(bc) + b(ca) + c(ab) = 0$ and $(ab)c + (bc)a + (ca)b = 0$
   (c) (4-member identity) $a(bc) - (ab)c$ is unchanged if we permute $a$ and $b$ (left-symmetric algebra)
   (d) (alternate 4-member identity) $a(bc) - (ab)c$ is unchanged if we permute $b$ and $c$ (right-symmetric algebra)
   (e) (6-member identity) $[a, bc] + [b, ca] + [c, ab] = 0$

2. Let $B$ be a bilinear form on $V$. Then
   $$ o_{V, B} = \{ a \in \mathfrak{gl}_V | B(a(u), v) + B(u, a(v)) = 0 \forall u, v \in V \} . $$
   
   (a) Show that for $a, b \in \mathfrak{gl}_n(\mathbb{F})$, $\text{tr}[a, b] = 0$.
   (b) Show that $o_{V, B}$ is a subalgebra of $\mathfrak{gl}_V$.

3. Show that
   $$ o_{\mathbb{F}^n, B} = \{ a | a^T B + B a = 0 \} . $$

4. If $f : \mathfrak{gl}_n(\mathbb{F}) \Rightarrow \mathbb{F}$ is a linear function such that $f((a, b)) = 0$ for any $a, b$, then $f = \lambda \cdot \text{tr}$ for some $\lambda \in \mathbb{F}$.

2 Lecture 2

1. A derivation of an arbitrary algebra is a vector space endomorphism $D$ of $A$ such that $D(ab) = D(a)b + aD(b)$. (This is called the Leibniz rule.) Let $\text{Der}(A)$ be the subspace in $\text{End} A$ of all derivations of $A$. Prove that $\text{Der}(A)$ is a subalgebra of $\mathfrak{gl}_A$ with the usual bracket.

2. Let $A = \mathbb{F}[x_1, x_2, \ldots, x_n]$. Define a bracket on $A$
   $$ \{ f, g \} = \sum_{i,j=1}^n \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \{ x_i, x_j \} $$
   for some choice of $\{ x_i, x_j \} \in A$. Prove that this is a Poisson bracket (i.e. the Lie algebra axioms hold) if and only if the skew symmetry axiom holds: $\{ x_i, x_i \} = 0$, $\{ x_i, x_j \} = -\{ x_j, x_i \}$, and the Jacobi identity holds for $x_i, x_j, x_k$.

3. Let $\phi : g_1 \to g_2$ be a homomorphism. Then show that
   (a) $\ker \phi$ is an ideal of $g_1$, 

(b) $\text{im } \phi$ is a subalgebra of $g_2$.
(c) $\text{im } \phi \cong g_1/\ker \phi$.

4. An algebraic group $G$ over a field $\mathbb{F}$ is a collection of polynomials $\{P_\alpha\}, \alpha \in I$, on the space of matrices $\text{Mat}_{n \times n}(\mathbb{F})$, such that for any unital commutative associative algebra $A$ over $\mathbb{F}$, the set

$$G(A) := \{g \in \text{Mat}_{n \times n}(A) \mid g \text{ non-singular}, P_\alpha(g) = 0 \forall \alpha \in I\}$$

is a group under the matrix multiplication.

Let $B \in \text{Mat}_{n \times n}(\mathbb{F})$, and let $O_{n,B}(A) = \{g \in \text{GL}_{n}(A) \mid g^TBg = B\}$. Show that this is an algebraic group.

5. The algebra of dual numbers is $D = \mathbb{F}[\epsilon]/(\epsilon^2) = \{a + b\epsilon | a, b \in \mathbb{F}\}$. The Lie algebra $\text{Lie } G$ of an algebraic group $G$ is $\text{Lie } G = \{X \in \text{gl}_n(\mathbb{F}) | I_n + \epsilon X \in G(D)\}$.

(a) $(I_n + \epsilon X)^{-1} = I_n - \epsilon X$.
(b) $\text{Lie } GL_n = \text{gl}_n(\mathbb{F})$, $\text{Lie } SL_n = \mathfrak{s}_n(\mathbb{F})$, $\text{Lie } O_{n,B} = \mathfrak{o}_{g^{\mathbb{F}},B}$.

3 Lecture 3

1. The center of a Lie algebra is $Z(g) = \{c \in g | [c, a] = 0 \forall a \in g\}$. Prove that $Z(\text{gl}_n(\mathbb{F})) = FI_n$, $Z(\mathfrak{s}_n(\mathbb{F})) = \mathbb{F}$ if $n$ does not divide char $\mathbb{F}$, and 0 otherwise.

2. For a finite dimensional Lie algebra $g$, $\dim Z(g) \neq \dim g - 1$.

3. $\dim Z(g) = \dim g - 2$ in exactly the following cases:

(a) $g = b \oplus \mathfrak{ab}_{n-2}$ where $b$ is a two-dimensional non-abelian Lie algebra, and $\mathfrak{ab}_m$ is an abelian Lie algebra with $m$ dimensions.

(b) $g = \text{Heis}_3 \oplus \mathfrak{ab}_{n-3}$, where $\text{Heis}_{2n+1}$ is the Lie algebra with basis $p_i, q_i, c$ for $1 \leq i \leq n$ and brackets $[p_i, q_i] = [q_i, p_i] = c$, with all other brackets 0.

4. For finite dimensional $V$, show that $A \in \text{End } V$ is nilpotent iff all eigenvalues are 0.

5. Engel’s Theorem states that if $g \subset \text{gl}_V$ is a finite-dimensional subalgebra consisting only of nilpotent isomorphisms (but not necessarily all of them) and $V$ is nonzero, then there exists $v \neq 0$ in $V$ that is killed by all endomorphisms in $g$. Deduce from Engel’s Theorem that if $\pi : g \Rightarrow \text{gl}_V$ is a Lie algebra representation of $g$ in $V$, for a finite dimensional $V$, then there exists a basis of $V$ in which all operators $\pi(a), a \in g$ have strictly upper triangulator matrices. Hint: $\dim \pi(g) \leq \dim \text{End } V \leq (\dim V)^2$.

6. It is important in Engel’s Theorem that $g$ is a subalgebra, not just a subspace. Show that $\mathbb{F} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \mathbb{F} \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ consists of nilpotent matrices, but there is no common eigenvector.