Twisting

(0.0.1) associated primes

Let \( M \) be a finite module over a noetherian ring \( A \), not the zero module. The annihilator \( \text{ann}(m) \) of an element \( m \) of \( M \) is the set of elements \( a \) of \( A \) such that \( am = 0 \). The annihilator is an ideal of \( A \). If the annihilator of an element \( m \) is a prime ideal \( P \), then \( P \) is called an associated prime of \( M \).

The set \( S \) of annihilators, ideals that are annihilators of nonzero elements of \( M \), is nonempty, and because 1 doesn’t annihilate any nonzero element of \( M \), \( S \) doesn’t contain the unit ideal. Since \( A \) is a noetherian ring, \( S \) contains at least one maximal member.

maxann

0.0.2. Lemma. Let \( M \) be a finite, nonzero module over a commutative ring \( A \), and let \( S \) be the set of annihilators of nonzero elements of \( M \).

(i) The maximal elements of \( S \) are associated primes of \( M \).

(ii) Suppose that the annihilator of an element \( m \) of \( M \) is a prime ideal \( P \), so that \( P \) is an associated prime. Let \( N \) be the submodule isomorphic to \( A/P \) that is generated by \( m \). The annihilator of any nonzero element of \( N \) is \( P \).

(iii) Let \( P_1, P_2, \ldots \) be distinct associated primes of \( M \). Let \( m_1, m_2, \ldots \) be elements of \( M \) such that \( P_i = \text{ann}(m_i) \), and let \( N_i \) be the submodule of \( M \) generated by \( m_i \). The submodules \( N_1, N_2, \ldots \) are independent, i.e., the sum \( N_1 + N_2 + \cdots \) is the direct sum \( N_1 \oplus N_2 + \cdots \).

(iv) The set of associated primes of \( M \) is finite and nonempty.

defMreg

proof. (i) Let \( P \) be a maximal annihilator, say the annihilator of \( m \in M \), and suppose that \( a, b \) are elements of \( A \) such that \( ab \) is in \( P \) but \( b \notin P \). So \( bm \neq 0 \). The annihilator of \( bm \) contains \( P \), and since \( P \) is maximal, it is equal to \( P \). Then since \( abm = 0, a \in P \).

(ii) Let \( x = bm \) be a nonzero element of \( N \), and let \( a \) be an element that annihilates \( bm \). Then \( abm = 0 \) so \( ab \in P \), but \( bm \neq 0 \) so \( b \notin P \). Therefore \( a \in P \). This shows that \( \text{ann}(bm) = P \).

(iii) We show that \( N_1 + \cdots + N_k \) are independent by induction on \( k \). The case \( k = 1 \) is trivial. For \( k > 1 \), we arrange indices so that \( P_k \not\subset P_1 \), and we let \( \alpha \) be an element of \( P_k \) that isn’t in \( P_1 \). Say that \( x_1 + \cdots + x_k = 0 \) with \( x_i \in N_i \). Let \( y_1 = \alpha x_1 \). So \( y_k = 0 \) and \( y_1 + \cdots + y_{k-1} = 0 \), but \( y_1 \neq 0 \). This contradicts the induction hypothesis that \( N_1, \ldots, N_{k-1} \) are independent.

(iv) Let \( N_1, N_2, \ldots \) be as in (iii), with \( P_i \) distinct. Then, since \( M \) is a finite module and \( A \) is noetherian, the strictly increasing sequence \( N_1 \subset (N_1 \oplus N_2) \subset (N_1 \oplus N_2 \oplus N_3) \subset \cdots \) must be finite. \( \square \)

locinject

0.0.3. Definition. We say that an element \( s \) of \( A \) is \( M \)-regular if \( M \) is \( s \)-torsion-free.

If \( s \) is \( M \)-regular, the map from \( M \) to its localization \( M_s \) will be injective.

0.0.4. Corollary. Let \( M \) be a finite \( A \)-module, and let \( s \) be an element of \( A \). Then \( s \) is \( M \)-regular if and only if \( s \) is not contained in any associated prime of \( M \).

proof. If \( s \) is contained in the associated prime \( P = \text{ann}(m) \), then \( sm = 0 \), so \( m \) is an \( s \)-torsion element. If \( s \) isn’t contained in any associated prime \( P \), then because maximal annihilators are associated primes, \( s \) isn’t in \( \text{ann}(m) \) for any nonzero element \( m \).

Note. Suppose that \( A \) is the coordinate algebra of the affine variety \( X \). The finite set \( \{P_1, \ldots, P_k\} \) of associated primes of \( M \) corresponds to a finite set of closed subvarieties \( \{Y_1, \ldots, Y_k\} \) of \( X = \text{Spec} A \). An element \( s \) has a zero locus \( V(s) \) in \( X \), and \( s \notin P_i \) if and only if \( V(s) \not\supset Y_i \). So \( M \) is \( s \)-torsion free if and only if \( V(s) \) doesn’t contain any of the subvarieties \( Y_1, \ldots, Y_k \). This is often easy to check.

maxanninloc

0.0.5. Proposition. Let \( M \) be a finite module over a noetherian domain \( A \) and let \( s \) be a nonzero element of \( A \). Also, let \( A' \) and \( M' \) denote the localizations \( A_s \) and \( M_s \), respectively. The associated primes of the \( A' \)-module \( M' \) are the ideals of the form \( P' = P_s \), where \( P \) is an associated prime of \( M \) such that \( s \notin P \).
proof. The prime ideals of \( A' \) have the form \( P' = P_s \), where \( P \) is a prime ideal of \( A \) that doesn’t contain \( s \). (If \( s \in P \), then \( P_s \) is the unit ideal of \( A_s \).)

Let \( m' = s^{-r} m \) be an element of \( M' \) whose annihilator in \( A' \) is the prime ideal \( P' \). Then \( P' \) is the localization of a prime ideal \( P \) of \( A \) that doesn’t contain \( s \). Since \( s \) is invertible in \( A' \), an element \( a' = s^{-k} a \) annihilates \( m' \) if and only if \( a \) annihilates the image of \( m \) in \( M_s \), and this is true if and only if \( a \) annihilates \( m \) in \( M \). So \( a' \) is in \( P' \) if and only if \( a \) is in \( P \). \( \square \)

(0.0.6) \( \mathcal{O} \)-modules (review)

As defined in class, an \( \mathcal{O} \)-module on a variety \( X \) is a map

\[
(\text{affine opens})^0 \xrightarrow{\mathcal{M}} (\text{modules})
\]

that associates an \( \mathcal{O}(U) \)-module \( \mathcal{M}(U) \) to every affine open set \( U \), and such that, if \( U_s \) is a localization of an affine open set \( U \), then \( \mathcal{M}(U_s) \) is the module \( \mathcal{M}(U)_s \) obtained by localizing \( \mathcal{M}(U) \). The sheaf property extends such a module uniquely to a functor

\[
(\text{opens})^0 \rightarrow (\text{modules})
\]

that we denote by \( \mathcal{M} \) too. To make this extension, one first shows that, if \( V \subset U \) is an inclusion of affine open sets, there is a natural module homomorphism \( \mathcal{M}(U) \rightarrow \mathcal{M}(V) \). Then if \( Y \) is any open set, we choose a covering of \( Y \) by affine open sets \( \{U_i\} \). The intersections \( U^j = U_i \cap U^j \) are also affine, and so there are maps \( \mathcal{M}(U^j) \rightarrow \mathcal{M}(U_i) \) and \( \mathcal{M}(U^j) \rightarrow \mathcal{M}(U^j) \). A section of \( M \) on \( Y \) is given by a collection \( m_i \) of sections on \( U^i \) such that the restrictions of \( m_i \) to \( U^j \) are equal, i.e., \( m_i = m_j \) on \( U^j \).

Let \( \mathcal{M} \) be a finite \( \mathcal{O} \)-module on a variety \( X \), let \( \{U_i\} \) be a covering of \( X \) by finitely many affine varieties \( U_i = \text{Spec} \ A_i \), and let \( M_i \) be the finite \( A_i \)-module \( \mathcal{M}(U_i) \). Each \( M_i \) has finitely many associated primes \( P_{iv} \) of \( A_i \), and the zero sets of these associated prime ideals are closed subsets of \( U_i \), call them \( Y_{iv} \). The closure \( Y_{iv} \) of \( Y_{iv} \) in \( X \) is a proper closed subset of \( X \). We put these closures together for all \( i \), obtaining a finite set \( Y_1, Y_2, ..., Y_N \) of proper closed subsets of \( X \).

(0.0.7) Proposition. With notation as above, let \( V = \text{Spec} \ A' \) be another affine open subset of \( X \), and let \( M' = \mathcal{M}(V) \). If \( s \) is a nonzero element of \( A' \) whose zero set in \( V \) doesn’t contain any of the sets \( Y_j \cap V \), then \( s \) is an \( M' \)-regular element of \( A' \), and therefore the map from \( M' \) to its localization \( M'_s \) is injective.

proof. We can cover \( V \) by open sets \( V^\nu \) that are closed sets, both of \( V \) and of one of the sets \( U_i \) \((3.3.21)\). Let \( M'_\nu \) be the corresponding \( A'_\nu \)-module The sheaf property for \( \mathcal{M} \) shows that an element \( s \) of \( A' \) is \( M' \)-regular if and only if it is \( M'_\nu \)-regular for every \( \nu \), and Proposition \((0.0.3)\) shows that this will be true if and only if the zero locus of \( s \) doesn’t contain any of sets \( Y_j \cap V \). \( \square \)

(0.0.8) generating a module

A set \( m = (m_1, ..., m_k) \) of global sections of an \( \mathcal{O} \)-module \( \mathcal{M} \) generates \( \mathcal{M} \) if the map

\[
\mathcal{O}^k \xrightarrow{m} \mathcal{M}
\]

that sends a section \( (\alpha_1, ..., \alpha_k) \) of \( \mathcal{O}^k \) on an open set \( U \) to the combination \( \sum \alpha_i m_i \) is surjective \((6.2.6)(iii))\). If the sections generate \( \mathcal{M} \), then they (more precisely, their restrictions) generate the \( \mathcal{O}(U) \)-module \( \mathcal{M}(U) \) for every affine open set \( U \). They may not generate when \( U \) isn’t affine.

(0.0.9) twisting an \( \mathcal{O} \)-module

To define the twists of an \( \mathcal{O} \)-module \( \mathcal{M} \) on a projective variety \( X \), we may extend by zero to obtain a module on the ambient projective space \( P = \mathbb{P}^d \), and twist this extension by zero. So we may assume that
the variety $X$ is projective space. The twist $\mathcal{M}(n)$ of the $\mathcal{O}$-module $\mathcal{M}$ is defined to be the tensor product $\mathcal{M} \otimes_\mathcal{O} \mathcal{N}$. If $U$ is an affine open set, then $[\mathcal{M} \otimes_\mathcal{O} \mathcal{N}](U) = \mathcal{M}(U) \otimes_{\mathcal{O}(U)} \mathcal{N}(U).

It is easy to describe the sections of the twist $\mathcal{M}(n)$ on the standard affine open sets $U^i$. On $U^0$, $\mathcal{O}(n)$ is a free module of rank 1, with basis $x_0^n$. Therefore a section of $\mathcal{M}(n)$ on $U^0$ can be written as $m \otimes ax_0^n$ for some regular function $a$ on $U^0$. And because the tensor product is over the structure sheaf $\mathcal{O}$, the coefficient $a$ can be moved across the tensor symbol: $m \otimes ax_0^n = am \otimes x_0^n$. Then this expression for a section on $U^0$ becomes unique.

**0.0.11. Corollary.** The sections of $\mathcal{M}(n)$ on the standard affine open set $U^0$ can be written uniquely in the form $m \otimes x_0^n$. □

**0.0.12. Theorem.** Let $\mathcal{M}$ be a finite $\mathcal{O}$-module on projective space $X$. Then for large $n$, $\mathcal{M}(n)$ is generated by its global sections.

**proof.** We’ll use the standard covering $\{U^i\}$ of projective space to compute global sections, and we suppose that the coordinates are in general position.

Let $u_{ij}$ denote the ratio $x_i/x_j$. Then $u_{ij}u_{jk} = u_{ik}$, $u_{ij}u_{ji} = 1$, and $u_{jj} = 1$. The coordinate algebra $A_j = \mathcal{O}(U^j)$ is the polynomial ring $\mathbb{C}[u_{0j}, ..., u_{nj}]$ in the $n$ variables that remain when one remembers that $u_{jj} = 1$. The intersection $U^{ij}$ can be obtained by localizing either $U^i$ or $U^j$: Its coordinate ring is $A_{ij} = A_j[u_{-1}^{-1}]$, and also $A_{ij} = A_i[u_{-1}^{-1}]$. Let $M_j = \mathcal{M}(U^j)$, and let $M_{ij}$ denote the sections of $\mathcal{M}$ on the intersections $U^{ij} = U^i \cap U^j$. Then $M_{ij}$ is the localization $M_j[u_{-1}^{-1}]$ of $M_j$, and also the the localization $M_i[u_{-1}^{-1}]$ of $M_i$. Similarly, if $M_{0ij}$ denotes the sections of $\mathcal{M}$ on the triple intersection $U^{0ij} = U^0 \cap U^i \cap U^j$, then $M_{0ij}$ is the localization $M_{ij}[u_{-1}^{-1}]$ of $M_{ij}$. The next lemma completes the proof. □

**0.0.13. Lemma.** When coordinates are in general position, the localization maps $M_j \to M_{ij}$ and $M_{ij} \to M_{0ij}$ will be injective.

**proof.** We start with the given coordinates. When we move coordinates to general position, Proposition 0.0.7 applies. The coordinate axes will not contain any of the closed subvarieties $Y_1, ..., Y_N$, so the ratios $u_{ij}$ will be $M_{ij}$-regular. □