Problems for Quiz 2

The quiz Friday, May 11, will consist of five of the following problems.

You will be expected to justify your assertions, and you will be graded partly on style.

1. Let $X$ be the affine surface in $\mathbb{A}^3$ defined by the equation $x_1^3 + x_1x_2x_3 + x_1x_3 + x_2^2 + x_3 = 0$, and let $\overline{X}$ be its closure in $\mathbb{P}^3$. Describe the intersection of $\overline{X}$ with the plane at infinity in $\mathbb{P}^3$.

2. Let $A \subset B$ be finite-type domains. Prove that if $B$ is a finite $A$-module, the inclusion induces a surjective map $\text{Spec } B \to \text{Spec } A$.

3. With coordinates $x_0,x_1,x_2$ in the plane $\mathbb{P}$ and $s_0,s_1,s_2$ in the dual plane $\mathbb{P}^*$, let $C$ be a smooth projective plane curve $f = 0$ in $\mathbb{P}$, where $f$ is an irreducible homogeneous polynomial in $x$. Let $\Gamma$ be the locus of pairs $(x,s)$ of $\mathbb{P} \times \mathbb{P}^*$ such that the line $s_0x_0 + s_1x_1 + s_2x_2 = 0$ is the tangent line to $C$ at $x$. Prove that $\Gamma$ is a Zariski closed subset of the product $\mathbb{P} \times \mathbb{P}^*$.

4. Let $x_0,x_1,x_2$ be the coordinate variables in the projective plane $X$. The function field $K$ of $X$ is the field of rational functions in the variables $u_1,u_2,u_i = x_i/x_0$. Let $f(u_1,u_2)$ and $g(u_1,u_2)$ be polynomials. Under what circumstances does the point $(1,f,g)$ with values in $K$ define a morphism $X \to \mathbb{P}^2$?

5. Prove that if a subset $S$ of a variety $X$ is constructible in the Zariski topology and is closed in the classical topology, then it is closed in the Zariski topology.

6. Let $X = \mathbb{P}^2$. What are the sections of the twisting module $\mathcal{O}_X(n)$ on the open complement of the line $\{x_1 + x_2 = 0\}$?

7. Let $A$ be a finite type algebra that satisfies the descending chain condition: Every strictly decreasing chain $I_1 > I_2 > \cdots$ of ideals of $A$ is finite. Prove that $A$ has finite dimension as a complex vector space.

8. Let $X = \mathbb{P}^2$, and let $U$ be the complement of the point $(1,0,0)$ in $X$. Determine the sections of the structure sheaf $\mathcal{O}_X$ on $U$.

9. Let $X = \mathbb{P}^1$. Are the $\mathcal{O}_X$-modules $\mathcal{O} \oplus \mathcal{O}$ and $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ isomorphic?

10. Let $Y$ be the surface in $\mathbb{P}^3$ defined by an irreducible polynomial of degree 5. Determine the dimensions of the cohomology groups $H^q(Y,\mathcal{O}_Y)$.