18.721 Assignment 4

This assignment is due Friday, March 4

1. Let $X$ be the plane curve $y^2 = x(x-1)^2$, and let $A = \mathbb{C}[x, y]/(y^2 - x(x-1)^2)$ be its coordinate algebra. Let’s use $x, y$ also to denote the residues of those elements in $A$.

(a) Points of the curve can be parametrized by a variable $t$. Use the lines $y = t(x - 1)$ to determine such a parametrization.

(b) Let $B = \mathbb{C}[t]$ and let $T$ be the affine line $\text{Spec} \mathbb{C}[t]$. The parametrization gives us an injective homomorphism $A \to B$. Describe the corresponding morphism $T \to X$.

(c) Let $s = x - 1$. Show that $X$ is covered by the two localizations $X_s = \text{Spec} A_s$ and $X_x = \text{Spec} A_x$, where $A_s = A[s^{-1}]$ and $A_x = A[x^{-1}]$.

2. (a locally principal ideal) Notation is as in the previous problem. The maximal ideal $M$ of $X$ at the point $p = (0, 0)$ is generated by the two elements $x, y$.

(a) Show that the localized ideal $M_s$ of $A_s$, the ideal of $A_s$ that is generated by $M$, is a principal ideal. Do the same for the localized ideal $M_x$.

(b) Using the ideal of $\mathbb{C}[t]$ that is generated by $M$, show that $M$ is not a principal ideal.

3. The cyclic group $< \sigma >$ of order $n$ operates on the polynomial ring $R = \mathbb{C}[x, y]$, by $\sigma(x) = \zeta x$ and $\sigma(y) = \zeta y$, $\zeta = e^{2\pi i/n}$. Let $A$ be the ring of invariants.

(a) Describe the invariant polynomials.

(b) Show that the polynomials $u_i = x^iy^{n-i}, i = 0, ..., n$, generate the ring $A$.

(c) Find generators for the ideal of relations among the generators $u_i$ (the kernel of the homomorphism from the polynomial ring $\mathbb{C}[y_0, ..., y_n]$ to $A$ that sends $y_i$ to $u_i$).

4. Let $A = \mathbb{C}[x_1, ..., x_2]$, and let $B = A[\alpha]$, where $\alpha$ is an element of the fraction field $\mathbb{C}(x)$ of $A$. Describe the fibres of the morphism $Y = \text{Spec} B \to \text{Spec} A = X$. 