Morphisms

0.1 Morphisms of Affine Varieties

Let $X = \text{Spec } A$ and $Y = \text{Spec } B$ be affine varieties. Morphisms from $Y$ to $X$ are the maps that are allowed in algebraic geometry. As we will see, they correspond to algebra homomorphisms $A \xrightarrow{\varphi} B$.

Recall that, if $X = \text{Spec } A$, the regular functions on $X$ are the functions that are defined by the elements of $A$, the function defined by an element $\alpha$ of $A$ being $\alpha(p) = \pi_p(\alpha)$, where $A \xrightarrow{\pi_p} C$ is the homomorphism that corresponds to $p$.

We first look at the case that $X$ is an affine space $\mathbb{A}^n$ whose coordinate algebra is the polynomial algebra, $\mathbb{C}[x_1, \ldots, x_m]$. The affine variety $Y = \text{Spec } B$ can be arbitrary. A morphism $Y \xrightarrow{U} X$ is defined to be evaluation of a set $\beta = (\beta_1, \ldots, \beta_m)$ of regular functions on $Y$, i.e., of elements of $B$. When the regular functions $\beta$ are given, the morphism $U$ sends a point $q$ of $Y$ to the point $(\beta_1(q), \ldots, \beta_m(q))$ of $\mathbb{A}^m$.

The elements $(\beta_1, \ldots, \beta_m)$ can also be used to define an algebra homomorphism $\mathbb{C}[x_1, \ldots, x_m] \xrightarrow{\varphi} B$, namely the one that evaluates a polynomial $f(x)$ at $\beta : \varphi(f(x_1, \ldots, x_m)) = f(\beta_1, \ldots, \beta_m)$.

Morphisms $Y \xrightarrow{u} \mathbb{A}^m$ and algebra homomorphisms $\mathbb{C}[x] \xrightarrow{\varphi} B$ are both defined by a set $(\beta_1, \ldots, \beta_m)$ of arbitrary elements of $B$. So morphisms $Y \rightarrow \mathbb{A}^m$ and algebra homomorphisms $\mathbb{C}[x] \rightarrow B$ correspond bijectively.

For example, let $X$ be the affine $x$-line $\mathbb{A}^1_x$, so that $A = \mathbb{C}[x]$, and let $Y$ be the space of $2 \times 2$ matrices, so that $B = \mathbb{C}[y_{11}, y_{12}, y_{21}, y_{22}]$. The determinant $d(y) = y_{11}y_{22} - y_{12}y_{21}$ defines a morphism $Y \rightarrow X$ that sends a matrix $q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$ to its determinant $q_{11}q_{22} - q_{12}q_{21}$. The corresponding algebra homomorphism $\mathbb{C}[x] \xrightarrow{\varphi} \mathbb{C}[y_{ij}]$ sends a polynomial $f(x)$ to $f(y_{11}y_{22} - y_{12}y_{21})$.

Now let $X = \text{Spec } A$ and $Y = \text{Spec } B$ be arbitrary affine varieties. We choose a presentation $A = \mathbb{C}[x_1, \ldots, x_m]/(f_1, \ldots, f_k)$ of $A$, so that $X$ becomes the closed subvariety $V(f)$ of affine space $\mathbb{A}^m$. Then we have a natural way to define a morphism $Y \xrightarrow{u} X$, namely as a morphism $Y \xrightarrow{U} \mathbb{A}^m$ whose image lies in $X$.

We ask: When is the image of the morphism $Y \xrightarrow{U} \mathbb{A}^m$ defined by a set $(\beta_1, \ldots, \beta_m)$ of elements of $B$ contained in $X$? Since $U(q) = (\beta_1(q), \ldots, \beta_m(q))$, and since $X$ is the locus of zeros of the polynomials $f$, the image of $Y$ will be contained in $X$ if and only if $f(\beta_1(q), \ldots, \beta_m(q)) = 0$ for every point $q$ of $Y$.

We look at the corresponding homomorphism $\mathbb{C}[x] \xrightarrow{\varphi} B$ that sends $f(x)$ to the element $f(\beta)$ of $B$. Then $f(\beta(q)) = 0$ for all points $q$ of $\text{Spec } B$ if and only if $f(\beta)$ is the zero element of $B$. (??) We can express this by saying that $\beta = (\beta_1, \ldots, \beta_m)$ is a solution of the equations $f(x) = 0$ in $B$. And if $\beta$ is such a solution, the map $\varphi$ defines a map $A \rightarrow B$.

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\begin{array}{ccc}
\mathbb{C}[x] & \longrightarrow & B \\
\downarrow & & \downarrow \\
A & \longrightarrow & B
\end{array}
\]

Thus we have proved

**0.1.1. Proposition.** Let $A = \mathbb{C}[x_1, \ldots, x_m]/(f_1, \ldots, f_k)$ and $B$ be finite-type domains, and let $X = \text{Spec } A$ and $Y = \text{Spec } B$. There are bijective correspondences between the following sets:

- solutions of the equations $f_i(x) = 0$ in $B$,
- algebra homomorphisms $A \xrightarrow{\varphi} B$,
- morphisms $Y \xrightarrow{u} X$,

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\Box
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to be continued