Solutions to Problem Set 9

1) Since M is Noetherian, it is generated by elements $m_1, ..., m_k$. The map:

$$R \longrightarrow M \oplus ... \oplus M, \qquad r \mapsto (rm_1, ..., rm_k)$$

has kernel equal to the annihilator of M. By our assumption that M is a faithful module, the above map is an injection. Therefore, R itself is a submodule of the Noetherian module $M^{\oplus k}$, and is hence Noetherian.

2) Say M has generators $m_1, ..., m_k$. Letting $M_i \subset M$ denote the submodule generated by $m_1, ..., m_i$ we construct a chain of submodules:

$$0 \subset M_1 \subset ... \subset M_k = M$$

where each M_i/M_{i-1} is generated by the single element m_i. Therefore, this module is cyclic, and therefore $M_i/M_{i-1} \cong R/J_i$ for some ideal $J_i \subset R$. Because M is an Artinian R–module, then so are all the quotients $M_i/M_{i-1} \cong R/J_i$ (Proposition 6.3). Therefore, the ring R/J_i does not contain infinite descending chains of ideals, and so it is an Artinian ring. In virtue of Theorem 8.5, each R/J_i is therefore a Noetherian ring, which implies that it is also a Noetherian R–module. Thus M_i/M_{i-1} are all Noetherian R–modules, and Proposition 6.3 again implies that M is a Noetherian R–module.

3) The quotient $R/\text{Ann}(M)$ is Noetherian, so it is enough to show that this quotient has dimension 0. Consider a composition series for M:

$$0 \subset M_k \subset ... \subset M_1 \subset M_0 = M$$

where each M_{i-1}/M_i is a simple module. Simple modules are clearly cyclic, i.e. generated by a single element, so we have $M_{i-1}/M_i \cong R/\mathfrak{m}_i$ for some ideal $\mathfrak{m}_i$. The quotient module is simple if and only if $\mathfrak{m}_i$ is a maximal ideal. It is easy to observe that:

$$\text{Ann}(M) \supset \mathfrak{m}_1 ... \mathfrak{m}_k$$

Assume $\mathfrak{p} \supset \text{Ann}(M)$ for some prime ideal $\mathfrak{p}$, and therefore the above inclusion implies that $\mathfrak{p} \supset \mathfrak{m}_i$ for some $i \in \{1, ..., k\}$. Since $\mathfrak{m}_i$ is a maximal ideal, this implies that $\mathfrak{p} = \mathfrak{m}_i$, hence any prime ideal containing $\text{Ann}(M)$ is maximal. This finishes the proof of the fact that $R/\text{Ann}(M)$ has dimension 0.

4) Let us consider any ideal $I \subset R[[x]]$ and prove that it is finitely generated. Consider the sets:

$$J_n = \{a \in R \text{ such that } \exists \text{ formal power series } ax^n + O(x^{n+1}) \in I\}$$

It is easy to see that each $J_n \subset R$ is an ideal, and that they form a chain:

$$J_0 \subset J_1 \subset ... \subset J_n \subset ...$$

Because R is Noetherian, we have $J_n = J_{n+1} = J_{n+2} = ...$ and all the ideals J_k are finitely generated. So we may assume:

$$J_k = (a_{k,1}, ..., a_{k,d}) \qquad \forall 1 \leq k \leq n$$

for certain elements $a_{k,i}$, which means that there exist power series:

$$f_{k,i}(x) = a_{k,i}x^k + O(x^{k+1}) \in I$$

I claim that $I = (..., f_{k,i}(x), ...)^{1 \leq i \leq d}_{1 \leq k \leq n}$. Indeed, take any $g(x) \in I$, and just like in the Hilbert basis theorem, there exists a combination of the $f_{k,i}(x)$ such that:

$$h(x) := g(x) - \sum_{k=0}^{n-1} \sum_{i=1}^{d} \text{constant} \cdot f_{k,i}(x) = ax^n + O(x^{n+1})$$

Since $h(x) \in I$, it is enough to show that $h(x)$ can be written as a combination of the power series $f_{n,i}(x)$. Then use $J_n = J_{n+1} = ...$ to write:

$$h(x) = \sum_{i=1}^{d} b_i \cdot f_{n,i}(x) + O(x^{n+1})$$

but the formal power series denoted $O(x^{n+1})$ is also in I, and so its lowest order term is in the ideal $J_{n+1} = J_n$. Therefore we may write:

$$h(x) = \sum_{i=1}^{d} (b_i + xb'_i) f_{n,i}(x) + O(x^{n+2})$$

Repeating the argument shows that we may inductively construct formal power series $c_i(x) = b_i + xb'_i + ...$ such that $h(x) = \sum_{i=1}^{d} c_i(x) f_{n,i}(x)$. This shows that $h(x)$ is generated by the $f_{n,i}(x)$, which establishes the finite generation of the ideal I.