Solutions to Problem Set 8

1) a) First of all, each R/I_s is a Noetherian R–module, because any chain of R–submodules are also R/I_s–submodules. Then the conclusion follows from the fact that the direct sum of k Noetherian modules is also Noetherian.

b) If $\cap_{s=1}^k I_s = 0$, then the projection map of R–modules $R \to \oplus_{s=1}^k R/I_s$ is injective. The fact that the codomain is a Noetherian R–module implies that the domain R is a Noetherian R–module. But this is equivalent to R being a Noetherian ring.

2) Suppose we have an ascending chain of ideals $I_1 \subset I_2 \subset ...$ of R. The chain of ideals $I_1 R' \subset I_2 R' \subset ...$ stabilizes because R' is Noetherian. But then so does the chain:

$$\phi(I_1 R') \subset \phi(I_2 R') \subset ...$$

and the conclusion follows from the observation that $\phi(IR') = I\phi(R') = IR = I$ for all ideals $I \subset R$.

3) The homomorphism ϕ gives us two chains of submodules of M:

$$\text{Ker } \phi \subset \text{Ker } \phi^2 \subset \text{Ker } \phi^3 \subset ...$$

$$\text{Im } \phi \supset \text{Im } \phi^2 \supset \text{Im } \phi^3 \supset ...$$

By the assumption, both of these chains eventually stabilize, so $K := \text{Ker } \phi^n$ and $I := \text{Im } \phi^n$ for $n \geq$ some big number N. The natural inclusions induce an R–module homomorphism:

$$K \oplus I \to M$$

To conclude the proof, we must show that this homomorphism is an iso.

- injectivity: assume $u \in K \cap I$. Then $\phi^N(v) = u$ for some $v \in M$, hence $\phi^{2N}(v) = \phi^N(u) = 0$ because $u \in K$. Therefore $v \in \text{Ker } \phi^{2N} = \text{Ker } \phi^N$ and hence $u = \phi^N(v) = 0$

- surjectivity: for any $u \in M$, consider $\phi^N(u) \in \text{Im } \phi^N = \text{Im } \phi^{2N}$ and therefore $\phi^N(u) = \phi^{2N}(v)$ for some $v \in M$. This implies that $u - \phi^N(v) \in \text{Ker } \phi^N$ and therefore $u \in I + K$.

4) This is called Cohen's theorem. Read a nice proof of it on page 8 of this document: http://www.math.uconn.edu/ kconrad/blurbs/zorn1.pdf