Solutions to Problem Set 6

1) If $I = \mathfrak{p}^2$ for a prime ideal $\mathfrak{p}$, then:

$$xy \in \mathfrak{p}^2 \Rightarrow xy \in \mathfrak{p} \Rightarrow x \in \mathfrak{p} \text{ or } y \in \mathfrak{p} \Rightarrow x^2 \in \mathfrak{p}^2 \text{ or } y^2 \in \mathfrak{p}^2$$

Then we must simply ensure that $\mathfrak{p}^2$ is not primary itself, which as we saw in class, holds for $\mathfrak{p} = (x, z)$ and $R = \mathbb{C}[x, y, z]/(xy - z^2)$.

2) a) Take for example $\mathfrak{q} = (x)$, which is primary because $R/(x) = \mathbb{C}[y, z]/z^2$ has the property that any zero-divisor is divisible by z and hence is nilpotent. Note that:

$$(x)/(x, z)^2 = (x)/(x^2, xz, z^2) = (x)/(x^2, xz, xy) = R/(x, y, z)$$

so we can define $\mathfrak{p} = (x, y, z)$, which is maximal and hence prime.

b) As we saw in class, we expect $\mathfrak{p}$ to be an associated prime of $(x, z)^2$. Indeed, we have:

$$(x, z)^2 = \mathfrak{q} \cap \mathfrak{p}^2 = (x) \cap (x, y, z)^2$$

Note that $(x, y, z)^2$ is primary because its radical is maximal.

3) Consider an arbitrary element $z = a + b\sqrt{n}$ with $a, b \in \mathbb{Q}$. We will assume $b \neq 0$, since integers are already integral over $\mathbb{Z}$. Then the minimal polynomial of z over $\mathbb{Q}$ is quadratic:

$$(z - a)^2 - b^2 n = 0$$

Suppose that z is integral over $\mathbb{Z}$, and let:

$$z^s + \sum_{i=0}^{s-1} z^i c_i = 0 \tag{1}$$

be an equation satisfied by z with all $c_i \in \mathbb{Z}$. We may assume the above equality is minimal over $\mathbb{Z}$. Since this equation also holds over $\mathbb{Q}$, we conclude that:

$$(z - a)^2 - b^2 n \quad \text{divides} \quad z^s + \sum_{i=0}^{s-1} z^i c_i$$

over $\mathbb{Q}$. But Gauss' Lemma, this contradicts the minimality of the equation (1) unless $s = 2$ and:

$$(z-a)^2 - b^2 n \in \mathbb{Z}[z] \quad \Longrightarrow \quad 2a \text{ and } a^2 - b^2 n \in \mathbb{Z}$$

This either implies $a, b \in \mathbb{Z}$ for arbitrary n, or $a, b \in \mathbb{Z} + \frac{1}{2}$ for $n \equiv 1$ modulo 4.

4) a) For any $r \in R$, consider the polynomial:

$$P(z) = \prod_{g \in G} (z - \phi_g(r))$$

Then r is integral over R^G because the above polynomial is monic, has $r = \phi_1(r)$ a root, and its coefficients are G–invariant elements:

$$\phi_h(P(z)) = \prod_{g \in G} (z - \phi_h \circ \phi_g(r)) = \prod_{g \in G} (z - \phi_{hg}(r)) = \prod_{g \in G} (z - \phi_g(r)) = P(z)$$

b) The fact that B_{f_i} is integral over A_{f_i} implies that for all $b \in B$ there exists a relation of the form:

$$\frac{b^s}{f_i^k} = \sum_{i=0}^{s-1} \frac{b^i a_i}{f_i^{k_i}}$$

for some natural numbers n, k, k_i and $a_0, ..., a_{s-1} \in A$. By the definition of localization, this implies that:

$$b^s f_i^{k'} = \sum_{i=0}^{s-1} b^i a'_i$$

for some $k' \in \mathbb{N}$ and elements $a'_0, ..., a'_{s-1} \in A$. Then the claim that B is integral over A follows from the above relations for all $i \in \{1, ..., n\}$ and the fact that:

$$1 = (f_1, ..., f_n) \quad \Longrightarrow \quad 1 = (f_1^d, ..., f_n^d)$$

for all natural numbers d. Indeed, we have:

$$1 = \sum f_i c_i \quad \Longrightarrow \quad 1 = \left(\sum_{i=1}^{n} f_i c_i \right)^{dn}$$

and every summand in the expansion of the right hand side will have some f_i raised to the power d.