Solutions to Problem Set 5

1) We know from class that:

$$R_f = R[y]/(yf-1) = \mathbb{C}[x_1,...,x_n,y]/(I+(yf-1))$$

so Y is the subset of points in $\mathbb{C}^{n+1}$ cut out by the equations in I, together with the extra equation $yf(x_1,...,x_n) = 1$. Any point $(x_1,...,x_n,y) \in Y$ is uniquely determined by its image $(x_1,...,x_n) \in X$, since the last coordinate is forced to be $y = f(x_1,...,x_n)^{-1}$, so we conclude that:

$$Y \cong X \cap \{f(x_1,...,x_n) \neq 0\}$$

The map of topological spaces:

$$\text{Spec } R_f \longrightarrow \text{Spec } R$$

given by sending a prime ideal in R_f to is contraction in R is injective (as is the case for all localizations). However, as we saw on last week's homework, the image of the above map consists of those prime ideals which do not contain the element f.

2) Let M be a finitely generated projective module. Then there exists a module K and an isomorphism $R^{\oplus n} \cong M \oplus K$ for some natural number n. Since then $K \cong R^{\oplus n}/M$, the module K also has n generators $k_1,...,k_n$. We can therefore write down a short exact sequence:

$$R^{\oplus n} \xrightarrow{f} R^{\oplus n} \xrightarrow{g} M \longrightarrow 0$$

where the map f sends $(0,...,0,1,0,...,0)$ to k_i, and the map g is the projection map $R^{\oplus n} \to R^{\oplus n}/K \cong M$. This proves that M is finitely presented. To prove that M is flat, we use the same argument as in class. If $A \hookrightarrow B$ is any injective map of R–modules, then:

$$A^{\oplus n} = A \otimes_R R^{\oplus n} \cong A \otimes_R M \oplus A \otimes_R K$$

$$B^{\oplus n} = B \otimes_R R^{\oplus n} \cong B \otimes_R M \oplus B \otimes_R K$$

Since the map $A^{\oplus n} \to B^{\oplus n}$ is still injective, so is the restricted map between the direct summands $A \otimes_R M \to B \otimes_R M$. This proves that M is flat.

Conversely, assume M is a finitely presented flat module. Claim:

a finitely presented flat module over a local ring is free

Indeed, note that this would prove that every localization of M is free, and hence M is locally free = projective. To prove the claim, assume R is a local ring and consider:

$$0 \longrightarrow K \longrightarrow R^{\oplus n} \xrightarrow{\phi} M \longrightarrow 0$$

where M is flat and K is finitely generated. Therefore, the elements $m_i = \phi(e_i)$ generate M, and so their images $\bar{m}_i$ modulo the maximal ideal $\mathfrak{m}$ generate $M/\mathfrak{m}M$ as an $R/\mathfrak{m}R$ vector space. We may assume that $\bar{m}_i$ actually form a basis of $M/\mathfrak{m}M$ as an $R/\mathfrak{m}R$ vector space, otherwise by Nakayama's lemma we could replace the system of generators m_i by a smaller one. Therefore, the map $\bar{\phi} : (R/mR)^{\oplus n} \to M/\mathfrak{m}M$ is an isomorphism. However:

$$\text{Ker } \bar{\phi} = K/\mathfrak{m}K \tag{1}$$

hence $K = \mathfrak{m}K$, hence $K = 0$ by the fact that it is finitely generated and Nakayama's Lemma. This implies $M \cong R^{\oplus n}$, modulo statement (1). This statement following from the more general claim, that if M is any flat module and:

$$0 \longrightarrow M' \longrightarrow M'' \longrightarrow M \longrightarrow 0$$

any exact sequence, then $M' \otimes N \hookrightarrow M'' \otimes N$ is injective for all modules N. I will let you prove this result, which essentially follows from Exercise 2.24 in the book (after you unpackage the definition of Tor and its symmetry).

3) Since the ideal $\mathfrak{m}$ is maximal, the quotient is a finite field extension of $\mathbb{F}$:

$$\mathbb{K} := \mathbb{F}[x_1, ..., x_n]/\mathfrak{m} \supset \mathbb{F} \tag{2}$$

Then the chain of intermediate fields:

$$\mathbb{K}_n \supset \mathbb{K}_{n-1} \supset ... \supset \mathbb{K}_1 \supset \mathbb{K}_0 = \mathbb{F}$$

consists of finite extensions, were $\mathbb{K}_i = \mathbb{K}_{i-1}(a_i)$ and $a_i = \bar{x}_i$ is the image of the i–th variable in (2). Since $\mathbb{K}_i/\mathbb{K}_{i-1}$ is finite, it is generated by a single irreducible polynomial:

$$g_i(a_n) = 0 \quad \text{where } g \text{ has coefficients in } \mathbb{K}_{i-1}$$

We may regard g_i as a polynomial $f_i(x_1, ..., x_n)$ with coefficients in $\mathbb{F}$.

4) Let us first prove the case $n = 1$, to illustrate what the problem is asking. Any maximal ideal $\mathfrak{m} \subset \mathbb{F}[x]$ is generated by an irreducible polynomial $f(x)$. Saying that $a \in \mathbb{K}$ is a $\mathbb{K}$–point of $V(\mathfrak{m})$ is the same thing as saying that $f(a) = 0$. Then the problem is saying that:

$$f(a) = f(b) = 0 \qquad \Leftrightarrow \qquad \exists\, \sigma \in \mathrm{Gal}(\mathbb{K}/\mathbb{F}) \text{ such that } b = \sigma(a)$$

which is one of the main properties of an extension being Galois. We will do the general case by induction on n, so assume the problem holds for $n-1$ and let us do it for n. The maximal ideal $\mathfrak{m}$ will be as in Problem 3, so in particular will contain an irreducible polynomial $f_1(x_1)$. Since both a_1, b_1 are roots of this polynomial, the preceding argument implies that there exists $\sigma' \in \mathrm{Gal}(\mathbb{K}/\mathbb{F})$ such that $a_1 = \sigma'(b_1)$. Let:

$$\mathbb{F} \subset \mathbb{L} \subset \mathbb{K}$$

be the subextension generated by a_1. Then $\mathbb{K}/\mathbb{L}$ is Galois, and the points:

$$(a_2, ..., a_n) \qquad \text{and} \qquad (\sigma'(b_2), ..., \sigma'(b_n))$$

both vanish on the ideal of functions:

$$(f_2(c, x_2), ..., f_n(c, x_2, ..., x_n)) \subset \mathbb{L}[x_2, ..., x_n]$$

where $c = a_1 = \sigma'(b_1)$. This ideal is maximal since $\mathbb{L} \cong \mathbb{F}[x_1]/(f_1(x_1))$, hence:

$$\frac{\mathbb{F}[x_1, ..., x_n]}{(f_1(x_1), f_2(x_1, x_2), ..., f_n(x_1, ..., x_n))} \overset{x_1 \mapsto c}{\cong} \frac{\mathbb{L}[x_2, ..., x_n]}{(f_2(c, x_2), ..., f_n(c, x_2, ..., x_n))}$$

so then we can apply the induction hypothesis to obtain an automorphism $\sigma'' \in \mathrm{Gal}(\mathbb{K}/\mathbb{L})$ such that $a_i = \sigma''(\sigma'(b_i))$ for all $i \in \{2, ..., n\}$. Then $\sigma = \sigma'' \circ \sigma'$ does the trick.