Solutions to Problem Set 4

1) Suppose R_f is finitely generated. Then all its generators will be of the form:

$$\frac{r_N}{f^N} + \frac{r_{n-1}}{f^{N-1}} + ... + \frac{r_1}{f} + r_0$$

for various $r_0, ..., r_N$ and some large enough N. But then $\frac{1}{f^{N+1}}$ cannot be written as a combination of such generators (otherwise f would be a unit) contradiction.

2) This is the generalization of the Koszul resolution. Set:

$$R^{\oplus a_1} = R^{\oplus n}$$

with the map:

$$R^{\oplus n} \to \mathfrak{m} \quad \text{given by} \quad (f_1, ..., f_n) \mapsto x_1 f_1 + ... + x_n f_n$$

The kernel of this map consists of all n-tuples of polynomials $(f_1, ..., f_n)$ such that $x_1 f_1 + ... + x_n f_n = 0$. Show that such a tuple can be written as $f_i = \sum_{j \neq i} x_j g_{ij}$ for some polynomials g_{ij} that satisfy $g_{ij} = -g_{ji}$. So we set:

$$R^{\oplus a_2} = R^{\oplus \binom{n}{2}}$$

with the map

$$R^{\oplus \binom{n}{2}} \to \mathfrak{m} \quad \text{given by} \quad (..., g_{ij}, ...)_{i \neq j} \mapsto (..., f_i = \sum_{j \neq i} g_{ij} x_j, ...)_{i \in \{1,...,n\}}$$

In general, we set $a_k = \binom{n}{k}$, and we think of $R^{\oplus a_k}$ as having a basis indexed by polynomials $g_{i_1 ..., i_k}$, which are antisymmetric when switching any two indices:

$$g_{i_1 ... i_a ... i_b ... i_k} = -g_{i_1 ..., i_b ... i_a ... i_k} \tag{1}$$

Then the map $R^{\oplus a_k} \stackrel{f_k}{\to} R^{\oplus a_{k-1}}$ is given by:

$$(..., g_{i_1 ..., i_k}, ...) \mapsto \left(..., f_{i_1 ... i_{k-1}} = \sum_{j=1}^{n} g_{i_1 ... i_{k-1} j} x_j, ... \right)$$

Note that the sum in fact only goes over $j \notin \{i_1, ..., i_{k-1}\}$, because of the antisymmetry property (1). It's easy to see that $f_k \circ f_{k+1} = 0$, which implies that Im $f_{k+1} \subset$ Ker f_k. To prove the opposite inclusion, let us take an element:

$$G = (..., g_{i_1...i_k}, ...) \in \text{Ker } f_k \tag{2}$$

We can prove that this element lies in Im f_{k+1} by induction, and the induction step is provided by the following statement: if $(i_1...i_k)$ is the leading coefficient of (2), then there exists some H such that $G + f_{k+1}(H)$ has leading coefficient larger than $(i_1...i_k)$ in lexicographic ordering. Here, the **leading coefficient** of G is the smallest (in lexicographic ordering) k–element subset $(i_1 < ... < i_k)$ such that $g_{i_1...i_k} \neq 0$. Since $f_k(G) = 0$, we have:

$$x_{i_k} g_{i_1...i_k} = -\sum_{a > i_k} x_a g_{i_1 i_2 ... i_{k-1} a}$$

The reason why the sum in the right hand side only goes over $a > i_k$ is that we assumed $(i_1...i_k)$ is the leading coefficient of G. From this property, we infer that there exist polynomials s_a such that:

$$g_{i_1...i_k} = -\sum_{a > i_k} x_a s_a$$

If we define:

$$H = \left(..., h_{i_1...i_{k+1}} = \begin{cases} s_a & \text{if } (i_1, ..., i_{k+1}) = (i_1, ..., i_k, a) \text{ for } a > i_k \\ 0 & \text{otherwise} \end{cases}, ... \right)$$

then $G + f_{k+1}(H)$ has leading coefficient strictly greater than $(i_1, ..., i_k)$.

3) a) The fact that $m_1, ..., m_k$ generate M implies that the map $f : R^{\oplus k} \to M$ is surjective. Because M is projective, this map splits, and therefore we have an isomorphism $R^{\oplus k} \cong M \oplus K$ (just like in class).

b) The above isomorphism means that we also have a surjection $R^{\oplus k} \twoheadrightarrow K$, so K is also finitely generated. Moreover, we can tensor the isomorphism with $R/\mathfrak{m}$ and obtain:

$$(R/\mathfrak{m})^{\oplus k} \cong M/\mathfrak{m}M \oplus K/\mathfrak{m}K$$

Since both $R/\mathfrak{m}$ and $M/\mathfrak{m}M$ are k–dimensional vector spaces over $R/\mathfrak{m}$ (since the $\bar{m}_1, ..., \bar{m}_k$ form a basis), we conclude that $K/\mathfrak{m}K = 0$, i.e. $\mathfrak{m}K = K$.

c) Since K is finitely generated and $\mathfrak{m}K = K$, Nakayama's Lemma implies that $K = 0$, and therefore $R^{\oplus k} \cong M$.

4) a) Points of Spec R_f are prime ideals $\mathfrak{p} \subset R_f$. By the classification of ideals in localizations, these are the same as prime ideals $\mathfrak{p} \subset R$ which do not contain f. We conclude that Spec R_f corresponds to the subset of Spec R consisting of those prime ideals which do not contain f, which is the complement of the closed set $V(f) \subset$ Spec R.

b) Points of Spec $R_\mathfrak{p}$ are prime ideals $\mathfrak{q} \subset R_\mathfrak{p}$. By the classification of ideals in localizations, these are the same as prime ideals $\mathfrak{q} \subset \mathfrak{p}$. This gives rise to an inclusion:

$$\text{Spec } R_\mathfrak{p} \overset{i}{\subset} \text{Spec } R$$

which we claim is a continuous map of topological spaces. To prove this, we must show that i^{-1} takes closed sets to closed sets. A closed set in Spec R consists of all primes which contain a given ideal I. Its preimage under i^{-1} consists of all primes which contain I, but which are contained in $\mathfrak{p}$. This set is closed (it's mostly empty unless $I \subset \mathfrak{p}$).