1) Consider a maximal ideal $m \subset R$ and let the residue field be $F = R/m$. The isomorphism $R^{\oplus m} \cong R^{\oplus n}$ is given by an $n \times m$ matrix with entries in $R$, which has a two–sided inverse. The cosets of this matrix modulo $m$ give us an invertible matrix:

$$F^{\oplus m} \cong F^{\oplus n}$$

Since $F$ is a field, the above is an isomorphism of $F$–vector spaces. It is well-known that this is only possible if $m = n$ (otherwise $m \times n$ matrices values in a field cannot be invertible).

2) Consider the composition:

$$M_1 \xrightarrow{\text{inclusion}} M_1 + M_2 \xrightarrow{\text{projection}} (M_1 + M_2)/M_2$$

It is surjective by the very definition of $M_1 + M_2$, and its kernel is $M_1 \cap M_2$. Then the result follows from the general statement that if $A \hookrightarrow B$ are modules such that both $A$ and $B/A$ are finitely generated, then so is $B$. To prove this, take a collection $a_1, ..., a_k$ which generates $A$, and a collection $b_1, ..., b_l \in B$ such that their classes mod $A$ generate $B/A$. Then anything in $B$ can be written as:

$$r_1b_1 + ... + r_lb_l \pmod{A}$$

for some $r_1, ..., r_l \in R$, hence anything in $B$ can be written as:

$$r_1b_1 + ... + r_lb_l + r'_1a_1 + ... + r'_ka_k$$

for some $r_1, ..., r_l, r'_1, ..., r'_k \in R$. This proves that $B$ is finitely generated.

3) a) If $I \not\cong R$, then let $r = \phi^{-1}(1)$ and we have $I = (r)$ is principal. Therefore, all that we need to show is that $I \not\cong R^{\oplus n}$ for $n > 1$. If this were the case, then we would have an injective map of $R$–modules:

$$R \oplus R \oplus ... \oplus R \xrightarrow{\phi} R$$

Let $r = \phi(1, 0, 0, ..., 0)$ and $r' = \phi(0, 1, 0, ..., 0)$. Since $\phi(-r', r, 0, ..., 0) = 0$, we contradict the injectivity of $\phi$. 


b) Take some \( f \in N \setminus M \), and consider the \( R \)-module homomorphism:

\[
R \longrightarrow N/M, \quad r \mapsto rf \mod M
\]

This homomorphism is surjective, otherwise its image would correspond to an intermediate submodule \( M \subset Z \subset N \). Therefore, \( N/M \cong R/I \) for some ideal \( I \). If the ideal \( I \) were not maximal, then \( R/I \) would have a proper submodule, hence \( N/M \) would have a proper submodule, hence there would exist some \( Z \) such that \( M \subset Z \subset N \).

4) The crucial thing is that \( \mathbb{K} = \mathbb{F}[x]/(x^3 - 2) \) and the polynomial \( (x^3 - 2) \) splits completely in \( \mathbb{F} \):

\[
\mathbb{K} = \frac{\mathbb{F}[x]}{(x - \sqrt[3]{2})(x - \sqrt[3]{2}\omega)(x - \sqrt[3]{2}\omega^2)}
\]

where \( \omega \) is a complex cube root of 1. Therefore, we have ring homomorphisms:

\[
\mathbb{K} \otimes_{\mathbb{F}} \mathbb{K} = \mathbb{K} \otimes_{\mathbb{F}} \frac{\mathbb{F}[x]}{(x - \sqrt[3]{2})(x - \sqrt[3]{2}\omega)(x - \sqrt[3]{2}\omega^2)} \longrightarrow \mathbb{K}[x] \otimes_{\mathbb{F}} \frac{\mathbb{K}[x]}{(x - \sqrt[3]{2})(x - \sqrt[3]{2}\omega)(x - \sqrt[3]{2}\omega^2)} \cong K \times K \times K
\]

The composition is surjective, and therefore also injective because it is a linear map map of \( \mathbb{F} \)-vector spaces of dimension 9, hence an isomorphism. All that remains to show is how to construct the isomorphism \( \psi \). This actually holds for any field \( \mathbb{K} \) and distinct elements \( z_1, ..., z_k \in \mathbb{K} \), in which case we claim that:

\[
\frac{\mathbb{K}[x]}{(x - z_1) \cdots (x - z_k)} \cong \mathbb{K} \times \cdots \times \mathbb{K}
\]

The map \( \psi \) given by sending \( f(x) \mapsto (f(z_1), ..., f(z_k)) \) is clearly an injective ring homomorphism, and hence surjective because both the domain and target are \( \mathbb{K} \)-vector spaces of the same dimension.