Solutions to Problem Set 3

1) Consider a maximal ideal $\mathfrak{m} \subset R$ and let the residue field be $\mathbb{F} = R/\mathfrak{m}$. The isomorphism $R^{\oplus m} \cong R^{\oplus n}$ is given by an $n \times m$ matrix with entries in R, which has a two–sided inverse. The cosets of this matrix modulo $\mathfrak{m}$ give us an invertible matrix:

$$\mathbb{F}^{\oplus m} \cong \mathbb{F}^{\oplus n}$$

Since $\mathbb{F}$ is a field, the above is an isomorphism of $\mathbb{F}$–vector spaces. It is well-known that this is only possible if $m = n$ (otherwise $m \times n$ matrices values in a field cannot be invertible).

2) Consider the composition:

$$M_1 \overset{\text{inclusion}}{\hookrightarrow} M_1 + M_2 \overset{\text{projection}}{\twoheadrightarrow} (M_1 + M_2)/M_2$$

It is surjective by the very definition of $M_1 + M_2$, and its kernel is $M_1 \cap M_2$. Then the result follows from the general statement that if $A \hookrightarrow B$ are modules such that both A and B/A are finitely generated, then so is B. To prove this, take a collection $a_1, ..., a_k$ which generates A, and a collection $b_1, ..., b_l \in B$ such that their classes mod A generate B/A. Then anything in B can be written as:

$$r_1 b_1 + ... + r_l b_l \text{ mod } A$$

for some $r_1, ..., r_l \in R$, hence anything in B can be written as:

$$r_1 b_1 + ... + r_l b_l + r'_1 a_1 + ... + r'_k a_k$$

for some $r_1, ..., r_l, r'_1, ..., r'_k \in R$. This proves that B is finitely generated.

3) a) If $I \overset{\phi}{\cong} R$, then let $r = \phi^{-1}(1)$ and we have $I = (r)$ is principal. Therefore, all that we need to show is that $I \ncong R^{\oplus n}$ for $n > 1$. If this were the case, then we would have an injective map of R–modules:

$$R \oplus R \oplus ... \oplus R \overset{\phi}{\hookrightarrow} R$$

Let $r = \phi(1, 0, 0, ..., 0)$ and $r' = \phi(0, 1, 0, ..., 0)$. Since $\phi(-r', r, 0, ..., 0) = 0$, we contradict the injectivity of ϕ.

b) Take some $f \in N \backslash M$, and consider the R–module homomorphism:

$$R \longrightarrow N/M, \qquad r \mapsto rf \bmod M$$

This homomorphism is surjective, otherwise its image would correspond to an intermediate submodule $M \subset Z \subset N$. Therefore, $N/M \cong R/I$ for some ideal I. If the ideal I were not maximal, then R/I would have a proper submodule, hence N/M would have a proper submodule, hence there would exist some Z such that $M \subset Z \subset N$.

4) The crucial thing is that $\mathbb{K} = \mathbb{F}[x]/(x^3-2)$ and the polynomial (x^3-2) splits completely in $\mathbb{F}$:

$$\mathbb{K} = \frac{\mathbb{F}[x]}{(x-\sqrt[3]{2})(x-\sqrt[3]{2}\omega)(x-\sqrt[3]{2}\omega^2)}$$

where ω is a complex cube root of 1. Therefore, we have ring homomorphisms:

$$\mathbb{K} \otimes_{\mathbb{F}} \mathbb{K} = \mathbb{K} \otimes_{\mathbb{F}} \frac{\mathbb{F}[x]}{(x-\sqrt[3]{2})(x-\sqrt[3]{2}\omega)(x-\sqrt[3]{2}\omega^2)} \twoheadrightarrow$$

$$\twoheadrightarrow \frac{\mathbb{K}[x]}{(x-\sqrt[3]{2})(x-\sqrt[3]{2}\omega)(x-\sqrt[3]{2}\omega^2)} \stackrel{\psi}{\cong} \mathbb{K} \times \mathbb{K} \times \mathbb{K}$$

The composition is surjective, and therefore also injective because it is a linear map map of $\mathbb{F}$–vector spaces of dimension 9, hence an isomorphism. All that remains to show is how to construct the isomorphism ψ. This actually holds for any field $\mathbb{K}$ and distinct elements $z_1, ..., z_k \in \mathbb{K}$, in which case we claim that:

$$\frac{\mathbb{K}[x]}{(x-z_1)...(x-z_k)} \stackrel{\psi}{\cong} \underbrace{\mathbb{K} \times ... \times \mathbb{K}}_{k \text{ times}}$$

The map ψ given by sending $f(x) \mapsto (f(z_1), ..., f(z_k))$ is clearly an injective ring homomorphism, and hence surjective because both the domain and target are $\mathbb{K}$–vector spaces of the same dimension.