**Solutions to Problem Set 2**

1) The fact that $1+x$ is a unit for all $x \in I$ implies that $I$ is contained in the Jacobson radical of $R$. But since $I$ is maximal, the Jacobson radical is contained in $I$. We conclude that $I$ coincides with the Jacobson radical, hence $R$ is local. As for the question at the end, the Jacobson radical is always an ideal with the given property (even in non-local rings).

2) Any ideal in $\mathbb{R}[x]$ is principal (since polynomial rings over fields are Euclidean domains, hence principal ideal domains) and is therefore generated by a single polynomial $f(x) \in \mathbb{R}[x]$. If this polynomial factors, the ideal it generates cannot be prime. Therefore, prime ideals can only correspond to irreducible polynomials, which in $\mathbb{R}[x]$ are just the linear and quadratic ones:

$$\mathfrak{p} = (x-a) \quad \text{or} \quad \mathfrak{p} = (x^2 - bx + c)$$

with $a, b, c \in \mathbb{R}$, $b^2 < 4c$. In the former case, the prime ideal is determined by the real number $a$. In the latter case, the prime ideal is determined by a pair of conjugate complex, non-real numbers. We conclude that the points of Spec $\mathbb{R}[x]$ are in bijection with the upper half complex plane. Finally, the Zariski closed subsets consist of those primes $\mathfrak{p} = (f)$ which contain a given collection of polynomials $\{f_\alpha\}$. This amounts to requiring $f | f_\alpha$ for all $\alpha$, i.e. $f | \gcd(f_\alpha)$. Since the gcd is also a polynomial, this amounts to requiring $f$ to be one of finitely many linear or quadratic polynomials. We conclude that the proper Zariski closed subsets are all the finite ones.

3) It is enough to prove the problem for $k = 2$ and then prove it by induction. Then we consider the ideal $(e_1, e_2)$ and claim that it matches the ideal $(e)$ where $e = e_1 + e_2 - e_1 e_2$. First of all, $e$ is an idempotent:

$$e(1-e) = (e_1 + e_2 - e_1 e_2)(1-e_1)(1-e_2) = 0$$

Clearly $(e) \subset (e_1, e_2)$. For the opposite inclusion, note that $e_1 = e_1^2 = e_1^2 + e_1(1-e_1)e_2 = e_1 e \in (e)$ and similarly $e_2 \in (e)$.

4) Fix a prime ideal $\mathfrak{p}$ and consider the set of all prime ideals contained in $\mathfrak{p}$, partially ordered by inclusion. We want to show that this set has a minimal

element, and using Zorn's lemma, we need to show that any chain has a lower bound. So consider a chain of ideals:

$$\{\mathfrak{p}_\alpha\} \quad \text{s.t.} \quad \mathfrak{p}_\alpha \subseteq \mathfrak{p} \ \forall \alpha \qquad \text{and} \qquad \mathfrak{p}_\alpha \subseteq \mathfrak{p}_\beta \text{ or } \mathfrak{p}_\beta \subseteq \mathfrak{p}_\alpha \ \forall \alpha, \beta$$

and consider the ideal $\mathfrak{r} = \bigcap_\alpha \mathfrak{p}_\alpha$. To show that it is the desired lower point, we must show that $\mathfrak{r}$ is prime. So consider any two elements $x, y \notin \mathfrak{r}$. Then there exist $\alpha, \beta$ such that $x \notin \mathfrak{p}_\alpha$ and $y \notin \mathfrak{p}_\beta$. Without loss of generality we may assume $\mathfrak{p}_\alpha \subset \mathfrak{p}_\beta$, hence $x, y \notin \mathfrak{p}_\alpha$, hence $xy \notin \mathfrak{p}_\alpha$, hence $xy \notin \mathfrak{r}$. This concludes the proof of $\mathfrak{r}$ being prime.