Solutions to Problem Set 2

1) The fact that $1 + x$ is a unit for all $x \in I$ implies that $I$ is contained in the Jacobson radical of $R$. But since $I$ is maximal, the Jacobson radical is contained in $I$. We conclude that $I$ coincides with the Jacobson radical, hence $R$ is local. As for the question at the end, the Jacobson radical is always an ideal with the given property (even in non-local rings).

2) Any ideal in $R[x]$ is principal (since polynomial rings over fields are Euclidean domains, hence principal ideal domains) and is therefore generated by a single polynomial $f(x) \in R[x]$. If this polynomial factors, the ideal it generates cannot be prime. Therefore, prime ideals can only correspond to irreducible polynomials, which in $R[x]$ are just the linear and quadratic ones:

$$p = (x - a) \quad \text{or} \quad p = (x^2 - bx + c)$$

with $a, b, c \in \mathbb{R}$, $b^2 < 4c$. In the former case, the prime ideal is determined by the real number $a$. In the latter case, the prime ideal is determined by a pair of conjugate complex, non-real numbers. We conclude that the points of Spec $\mathbb{R}[x]$ are in bijection with the upper half complex plane. Finally, the Zariski closed subsets consist of those primes $p = (f)$ which contain a given collection of polynomials $\{f_\alpha\}$. This amounts to requiring $f|f_\alpha$ for all $\alpha$, i.e. $f|\text{gcd}(f_\alpha)$. Since the gcd is also a polynomial, this amounts to requiring $f$ to be one of finitely many linear or quadratic polynomials. We conclude that the proper Zariski closed subsets are all the finite ones.

3) It is enough to prove the problem for $k = 2$ and then prove it by induction. Then we consider the ideal $(e_1, e_2)$ and claim that it matches the ideal $(e)$ where $e = e_1 + e_2 - e_1e_2$. First of all, $e$ is an idempotent:

$$e(1 - e) = (e_1 + e_2 - e_1e_2)(1 - e_1)(1 - e_2) = 0$$

Clearly $(e) \subset (e_1, e_2)$. For the opposite inclusion, note that $e_1 = e_1^2 = e_1^2 + e_1(1 - e_1)e_2 = e_1e \in (e)$ and similarly $e_2 \in (e)$.

4) Fix a prime ideal $p$ and consider the set of all prime ideals contained in $p$, partially ordered by inclusion. We want to show that this set has a minimal
element, and using Zorn’s lemma, we need to show that any chain has a lower bound. So consider a chain of ideals:

\[
\{p_\alpha\} \quad \text{s.t.} \quad p_\alpha \subseteq p \quad \forall \alpha \quad \text{and} \quad p_\alpha \subseteq p_\beta \text{ or } p_\beta \subseteq p_\alpha \quad \forall \alpha, \beta
\]

and consider the ideal \( r = \bigcap_\alpha p_\alpha \). To show that it is the desired lower point, we must show that \( r \) is prime. So consider any two elements \( x, y \notin r \). Then there exist \( \alpha, \beta \) such that \( x \notin p_\alpha \) and \( y \notin p_\beta \). Without loss of generality we may assume \( p_\alpha \subset p_\beta \), hence \( x, y \notin p_\alpha \), hence \( xy \notin p_\alpha \), hence \( xy \notin r \). This concludes the proof of \( r \) being prime.