Solutions to Problem Set 11

1) The reason we assume R/I is Artinian is to be able to write:

$$\chi_M^I(n) = \text{length}_R(M/I^n M) = \sum_{i=1}^{n} \text{length}_R(I^{i-1}M/I^i M) \qquad (1)$$

This is nice, because the modules $I^{i-1}M/I^i M$ are defined over the Artinian ring R/I, and hence have finite length (since both I and M are finitely generated, and finitely generated modules over Artinian rings are both Artinian and Noetherian), so (1) is well-defined. The problem therefore reduces to the following claim:

If A is an Artinian ring and N a finitely generated module with support $\{\mathfrak{p}_1, ..., \mathfrak{p}_k\}$, then:

$$\text{length}(N) = \sum_{i=1}^{k} \text{length}(N_{(\mathfrak{p}_i)}) \qquad (2)$$

(indeed, the above statement for $A = R/I$ and $N = I^{i-1}M/I^i M$ solves our problem). We may actually assume that $\mathfrak{p}_1, ..., \mathfrak{p}_k$ are all the prime ideals of A, since if such a prime is not in the support, then it contributes nothing to the right hand side of (2). Then we will use the structure theorem of Artinian rings to infer that:

$$A \cong \prod_{i=1}^{k} A_{(\mathfrak{p}_i)}$$

and with respect to this decomposition we have an isomorphism of modules:

$$N \cong \prod_{i=1}^{k} N_{(\mathfrak{p}_i)}$$

With this in mind, (2) reduces to the fact that the length of a module $N_1 \times N_2$ over the ring $A_1 \times A_2$ is equal to the length of $N_1 \curvearrowleft A_1$ plus the length of

$N_2 \curvearrowleft A_2$. This is clear, because if $... \subset N_1^i \subset ...$ is a composition series of N_1 and $... \subset N_2^j \subset ...$ is a composition series of N_2, then:

$$... \subset N_1^i \times \{0\} \subset ... \subset N_1 \times \{0\} \subset ... \subset N_1 \times N_2^j \subset ...$$

is a composition series of $N_1 \times N_2$.

2) The Artin-Rees Lemma implies that for n large enough:

$$I^n M \cap N = I(I^{n-1} M \cap N) \quad \Longrightarrow \quad N = IN$$

Then use Corollary 2.5 in the book.

3) Assume that R is a UFD, and consider any height 1 prime ideal $\mathfrak{p}$. Take a non-zero element $r \in \mathfrak{p}$, which by assumption admits a unique factorization into prime elements. Since $\mathfrak{p}$ is prime, it will contain one of these prime elements, say $p \in \mathfrak{p}$. But this implies that $(p) \subset \mathfrak{p}$, and since $\mathfrak{p}$ has height 1, we must have $(p) = \mathfrak{p}$.

Now assume any height 1 prime is principal, and take any irreducible element $r \in R$. Consider a minimal prime $\mathfrak{p}$ associated to the principal ideal (r), and by the Krull principal ideal theorem, $\mathfrak{p}$ must have height 1. Therefore, $\mathfrak{p}$ is principal, say $\mathfrak{p} = (p)$ for a prime element p. We therefore have $r = pu$, and the fact that r is irreducible implies that u must be a unit. Hence r is prime.

4) Consider the short exact sequence:

$$0 \longrightarrow \mathbb{C}[x_1, ..., x_k] \xrightarrow{\cdot f} \mathbb{C}[x_1, ..., x_k] \longrightarrow \mathbb{C}[x_1, ..., x_k]/(f) \longrightarrow 0$$

Since f has homogeneous degree d, this short exact sequence descends to an exact sequence of the graded pieces as follows:

$$0 \longrightarrow \mathbb{C}[x_1, ..., x_k]_{n-d} \xrightarrow{\cdot f} \mathbb{C}[x_1, ..., x_k]_n \longrightarrow (\mathbb{C}[x_1, ..., x_k]/(f))_n \longrightarrow 0$$

This implies that the dimension of the rightmost term equals:

$$\dim \mathbb{C}[x_1, ..., x_k]_n - \dim \mathbb{C}[x_1, ..., x_k]_{n-d} = \binom{n+k-1}{k-1} - \binom{n-d+k-1}{k-1}$$

so in particular it has degree $k-1$.