Solutions to Problem Set 1

1) All statements can be proved by induction in n, so it's enough to prove the case $n = 1$. So take a polynomial:

$$f(x) = c_0 + c_1 x + ... + c_k x^k$$

Remember that the sum of nilotents is nilpotent. Hence if all the c_i are nilpotent, then f is nilpotent. Conversely, if f is nilpotent, then c_0 is nilpotent. Hence $f - c_0$ is nilpotent, and repeating the argument for the polynomial $c_1 + c_2 x + ... + c_k x^{k-1}$ we infer that c_1 is nilpotent etc. The second bullet is proved analogously, once we recall that the sum of a nilpotent and a unit is a unit.

Now take a power series:

$$f(x) = c_0 + c_1 x + ... + c_k x^k + ...$$

If f is nilpotent, then c_0 must be nilpotent, as above. But then $f - c_0$ is nilpotent, so we can repeat the argument for the power series $c_1 + c_2 x + ...$ to infer that c_1 is nilpotent etc. To prove the fourth bullet, we use the identity:

$$\frac{1}{f(x)} = \frac{1}{c_0 + c_1 x + ...} = \sum_{n=0}^{\infty} \frac{(-c_1 x - c_2 x^2 - ...)^n}{c_0^{n+1}}$$

since the right hand side is a well-defined power series (only finitely many summands contribute to each x coefficient.

2) If $e \in R$ is an idempotent, then we may define $R_1 = R/(e)$ and $R_2 = R/(1-e)$ and consider the projection:

$$R \to R/(e) \times R/(1-e), \qquad a \mapsto (a \text{ mod } e, a \text{ mod } 1-e)$$

We claim that this is an isomorphism.

- injectivity: if $a \in (e) \cap (1-e)$, then $a = eb$ and $a = (1-e)c$ for some $b, c \in R$. This implies that $a = eb = e^2 b = ea = e(1-e)c = 0$

- surjectivity: we must show that for any $b, c \in R$ there exists an $a \in R$ such that $a \equiv b \bmod e$ and $a \equiv c \bmod 1-e$. Just define $a = b(1-e)+ce$.

To establish the opposite correspondence, to any decomposition $R = R_1 \times R_2$ we associate the element $(0,1) \in R$, which satisfies $(0,1)^2 = (0^2, 1^2) = (0,1)$, and hence is an idempotent.