Problem Set 8

Due: Tue, November 8 at 11 AM in the pset boxes outside room 4–174

NO COLLABORATION

1) Let R be a ring and $I_1, ..., I_k \subset R$ be ideals such that R/I_s is a Noetherian ring for all $s \in \{1, ..., k\}$. Prove that:

a) $\oplus_{s=1}^{k} R/I_s$ is a Noetherian R–module.

b) if $\cap_{s=1}^{k} I_s = \{0\}$, then R is a Noetherian ring.

2) Consider rings $R \subset R'$ together with a homomorphism of R–modules $\phi : R' \to R$ such that $\phi(1) = 1$ (such a homomorphism is called a "retraction"). Show that R' = Noetherian ring implies that R = Noetherian ring.

3) Let M be an R–module which is both Noetherian and Artinian. If $\phi : M \to M$ is an R–module homomorphism, then for large enough n prove that we have a direct sum decomposition of R–modules:

$$M = \text{Ker } \phi^n \oplus \text{Im } \phi^n$$

4) If all the prime ideals of a ring R are finitely generated, then R is Noetherian (Hint: use Zorn's lemma. The problem is pretty hard, so even if you do get stuck, write your argument as far as you can. You will get partial credit)