Problem Set 6

Due: Tue, October 25 at 11 AM in the pset boxes outside room 4–174

NO COLLABORATION

1) Find an example of a ring R and an ideal I which is not primary, but has the property that $xy \in I$ implies $x^n \in I$ or $y^n \in I$ for some n.

2) Consider the ring $R = \mathbb{C}[x, y, z]/(xy - z^2)$.

a) Construct a primary ideal $\mathfrak{q}$ such that $(x, z)^2 \subset \mathfrak{q} \subset R$ and:

$$\mathfrak{q}/(x, z)^2 = R/\mathfrak{p}$$

for some prime ideal $\mathfrak{p}$.

b) Find the primary decomposition of the ideal $(x, z)^2$ in R.

3) For a square free integer n (can also be negative) consider the field $\mathbb{Q}(\sqrt{n}) = \{a + b\sqrt{n}, a, b \in \mathbb{Q}\}$. Depending on n, describe all elements of this field which are integral over $\mathbb{Z}$.

4) a) Consider a commutative ring R together with an action of a finite group $G \curvearrowright R$ by automorphisms (this means that for all $g \in G$ we are given a ring homomorphism $\phi_g : R \to R$ such that $\phi_1 =$ identity and $\phi_{gh} = \phi_g \circ \phi_h$ for all $g, h \in G$). Show that R is integral over the ring of invariants $R^G = \{r \text{ such that } \phi_g(r) = r \ \forall g\}$.

b) Prove that being integral is a local property, i.e. if $A \subset B$ are arbitrary commutative rings and $f_1, ..., f_n \in A$ are such that $1 = (f_1, ..., f_n)$ and B_{f_i} is integral over A_{f_i} for all $i \in \{1, ..., n\}$, then B is integral over A.