Problem Set 5

Due: Tue, October 18 at 11 AM in the pset boxes outside room 4–174

NO COLLABORATION

1) Remember that an algebraic variety $X \subset \mathbb{C}^n$ is the subset of points cut out by some ideal of equations $I \subset \mathbb{C}[x_1, ..., x_n]$. In this case, we refer to $R = \mathbb{C}[x_1, ..., x_n]/I$ as the coordinate ring of the variety X.

Show that for any non-nilpotent element $f \in R$, the localization R_f is also the coordinate ring of some algebraic variety Y, and describe the connection between Y and X in terms of both geometry (i.e. as sets of points) and topology (more specifically the Zariski topology).

2) A module $M \curvearrowleft R$ is called **finitely presented** if there exist natural numbers m and n, and an exact sequence:

$$R^{\oplus m} \xrightarrow{f} R^{\oplus n} \xrightarrow{g} M \longrightarrow 0$$

In other words, not only does M have a finite number n of generators (the images of the standard basis under g), but there are only finitely many relations between these generators (the image of the standard basis onder f).

Show that any finitely generated projective module is also a finitely presented flat module, and vice versa.

3) For an arbitrary field $\mathbb{F}$, prove that any maximal ideal $\mathfrak{m} \subset \mathbb{F}[x_1, ..., x_n]$ is of the form:

$$\mathfrak{m} = (f_1(x_1), f_2(x_1, x_2), f_3(x_1, x_2, x_3), ..., f_n(x_1, ..., x_n))$$

where each $f_k(x_1, ..., x_k)$ is irreducible over $\mathbb{F}(x_1, ..., x_{k-1})$. You may use the fact (which we will prove later) that field extensions $\mathbb{K}/\mathbb{F}$ such that $\mathbb{K}$ is finitely generated as a $\mathbb{F}$–algebra are finite.

4) For any field extension $\mathbb{F} \subset \mathbb{K}$ and any ideal $\mathfrak{m} \subset \mathbb{F}[x_1, ..., x_n]$, we say that $(a_1, ..., a_n) \in \mathbb{K}^n$ is a $\mathbb{K}$–valued point of $V(\mathfrak{m})$ if all the polynomials in $\mathfrak{m}$ vanish at this point.

Show that if the extension is Galois and the ideal $\mathfrak{m}$ is maximal, then for any two $\mathbb{K}$–valued points $(a_1, ..., a_n)$ and $(b_1, ..., b_n)$ of $V(\mathfrak{m})$, there exists an element of the Galois group $\sigma \in \mathrm{Gal}(\mathbb{K}/\mathbb{F})$ such that:

$$(b_1, ..., b_n) = (\sigma(a_1), ..., \sigma(a_n))$$

(50 % of points for the case $n = 1$)