Problem Set 4

Due: Tue, October 4 at 11 AM in the pset boxes outside room 4–174

NO COLLABORATION

1) If R is an integral domain and $f \in R \backslash 0$ is not a unit, prove that R_f is not a finitely generated R–module.

2) Let $R = \mathbb{C}[x_1, ..., x_n]$ and let $\mathfrak{m} = (x_1, ..., x_n) \subset R$ denote a maximal ideal. Construct an exact sequence of R–modules:

$$0 \to R^{\oplus a_n} \to R^{\oplus a_{n-1}} \to ... \to R^{\oplus a_1} \to \mathfrak{m} \to 0$$

i.e. construct the numbers $a_n, ..., a_1$, together with R–module homomorphisms such that the kernel of each arrow is the image of the preceding one. Hint: do first the case $n = 3$ (worth 50% of the points for the problem).

3) Let R be a local ring, and $\mathfrak{m}$ be its unique maximal ideal. Consider any finitely generated projective R–module M:

a) Consider a set of generators $m_1, ..., m_k \in M$ such that their cosets $\bar{m}_1, ..., \bar{m}_k \in M/\mathfrak{m}M$ form a BASIS of the $R/\mathfrak{m}$–vector space $M/\mathfrak{m}M$. Show that these generators give rise to a surjection $f : R^{\oplus k} \twoheadrightarrow M$ and prove that:

$$R^{\oplus k} \cong M \oplus K$$

where $K = \text{Ker } f$.

b) Show that K is finitely generated and has the property that $\mathfrak{m}K = K$.

c) Conclude that M is a free module.

4) a) Let $f \in R$ be a non-zero divisor. Describe the topological space Spec R_f in relation to Spec R

b) Let $\mathfrak{p} \subset R$ be a prime ideal. Describe the topological space Spec $R_{\mathfrak{p}}$ in relation to Spec R