Problem Set 3

Due: Tue, September 27 at 11 AM in the pset boxes outside room 4–174

NO COLLABORATION

1) For a ring R, prove that $R^{\oplus m} \cong R^{\oplus n}$ as R–modules only if $m = n$.

2) Consider a ring R with a module N, and two submodules $M_1, M_2 \subset N$. We will write $M_1 + M_2$ for the submodule of N generated by M_1 and M_2 (i.e. the smallest submodule of N containing M_1 and M_2). Show that if $M_1 + M_2$ and $M_1 \cap M_2$ are finitely generated R–modules, then so are M_1 and M_2.

3) a) Prove that if an ideal $I \subset R$ is a free R–module, then it is principal.

b) Prove that if R–modules $M \subset N$ are such that there does not exist any intermediary module $M \subset Z \subset N$, then there exists an isomorphism of R–modules $N/M \cong R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$.

4) Let $\mathbb{F} = \mathbb{Q}[e^{\frac{2\pi i}{3}}]$ and $\mathbb{K} = \mathbb{F}[\sqrt[3]{2}]$. Construct (with proof) a ring isomorphism:

$$\mathbb{K} \otimes_{\mathbb{F}} \mathbb{K} \cong \mathbb{K} \times \mathbb{K} \times \mathbb{K}$$