Problem Set 11

Due: Tue, December 13 at 11 AM in the pset boxes outside room 4–174

NO COLLABORATION

1) The definition of the Hilbert-Samuel polynomial makes sense for any Noetherian ring R (not just local), any ideal $I \subset R$ such that the quotient R/I is Artinian, and any finitely generated module $M \curvearrowleft R$. Prove that if:

$$V(I) \cap \mathrm{Supp}(M) = \{\mathfrak{p}_1, ..., \mathfrak{p}_k\}$$

then the Hilbert-Samuel function satisfies:

$$\chi_M^I = \sum_{i=1}^{k} \chi_{M_{(\mathfrak{p}_i)}}^{I_{(\mathfrak{p}_i)}}$$

As before, we define $\chi_M^I(n) = \mathrm{length}_R(M/I^n M)$.

2) Consider a Noetherian ring R, an ideal $I \subset R$, and a finitely generated R–module M. If:

$$N = \bigcap_{n=0}^{\infty} I^n M$$

prove that there exists $r \in I$ such that $rN = N$.

3) Let R be a Noetherian domain. Prove that R is a UFD (i.e. every irreducible element of R is prime) if and only if every height 1 prime ideal of R is principal.

4) Let $f \in \mathbb{C}[x_1, ..., x_k]$ be a homogeneous polynomial of degree d. Compute the Hilbert polynomial of the graded ring $\mathbb{C}[x_1, ..., x_k]/(f)$ explicitly.