Problem Set 10

Due: Tue, November 29 at 11 AM in the pset boxes outside room 4–174

NO COLLABORATION

1) Consider an irreducible polynomial $f \in \mathbb{C}[x, y]$ of the form:

$$f(x, y) = ax + by + \text{quadratic and higher degree terms}$$

Consider the ring $R = \mathbb{C}[x, y]/(f)$ and the prime ideal $\mathfrak{p} = (x, y)/(f)$. Prove that the localization $R_{\mathfrak{p}}$ is a discrete valuation ring if and only if $(a, b) \neq (0, 0)$.

Note: the condition $(a, b) \neq (0, 0)$ is equivalent with saying that the curve $f(x, y) = 0$ is smooth at the point $(0, 0)$. If you want, you may use the fact that the Krull dimension of $\mathbb{C}[x, y]$ is 2.

2) Let R be a valuation ring, $\mathfrak{m}$ its maximal ideal, and $\mathfrak{p} \subset \mathfrak{m}$ some prime ideal. Show that there exists a valuation ring $R' \supset R$ whose maximal ideal is $\mathfrak{p}$, and such that $R/\mathfrak{p}$ is a valuation ring of the field $R'/\mathfrak{p}$.

3) Suppose we have a ring $\mathbb{C}[x] \subset R \subset \mathbb{C}[[x]]$ which is local with maximal ideal generated by x. Prove that R is a discrete valuation ring.

4) Compute the ideal class group of the ring of integers of $\mathbb{Q}(\sqrt{-5})$. Feel free to use the internet for this one, think of it as a little research project.