Problem Set 1

Due: Tue, September 13 at 11 AM in the pset boxes outside room 4–174

NO COLLABORATION

1) For any ring R, consider the polynomial ring in n variables $R[x_1, ..., x_n]$ and take any element:

$$f(x_1, ..., x_n) = \sum_{k_1, ..., k_n = 0}^{\text{some arbitrary natural number}} c_{k_1, ..., k_n} x_1^{k_1} ... x_n^{k_n}$$

for various coefficients $c_{k_1, ..., k_n} \in R$.

- prove that f is nilpotent if and only if all the $c_{k_1, ..., k_n}$ are nilpotent
- prove that f is a unit if and only if $c_{0, ..., 0}$ is a unit in R and all the other $c_{k_1, ..., k_n}$ are nilpotent

Now consider the ring of formal power series $R[[x_1, ..., x_n]]$, whose elements are by definition formal infinite sums:

$$f(x_1, ..., x_n) = \sum_{k_1, ..., k_n = 0}^{\infty} c_{k_1, ..., k_n} x_1^{k_1} ... x_n^{k_n}$$

In this ring, show that:

- prove that f is nilpotent only if all the $c_{k_1, ..., k_n}$ are nilpotent (explain why the "if" statement fails)
- prove that f is a unit if and only if $c_{0, ..., 0}$ is a unit

2) In any ring R, an element $e \in R$ is called an **idempotent** if $e^2 = e$. Given two rings R_1 and R_2, the **product ring** $R_1 \times R_2$ is defined with component–wise addition and multiplication (also set $0 = (0, 0)$ and $1 = (1, 1)$).

Fix R commutative. To any idemponent of R, associate a decomposition:

$$R = R_1 \times R_2$$

and to every such decomposition, associate an idempotent of R.