Problem Set 1

Due: Tue, September 13 at 11 AM in the pset boxes outside room 4–174

NO COLLABORATION

1) For any ring $R$, consider the polynomial ring in $n$ variables $R[x_1, ..., x_n]$ and take any element:

\[ f(x_1, ..., x_n) = \sum_{k_1, ..., k_n=0}^{\text{some arbitrary natural number}} c_{k_1, ..., k_n} x_1^{k_1} ... x_n^{k_n} \]

for various coefficients $c_{k_1, ..., k_n} \in R$.

- prove that $f$ is nilpotent if and only if all the $c_{k_1, ..., k_n}$ are nilpotent
- prove that $f$ is a unit if and only if $c_{0, ..., 0}$ is a unit in $R$ and all the other $c_{k_1, ..., k_n}$ are nilpotent

Now consider the ring of formal power series $R[[x_1, ..., x_n]]$, whose elements are by definition formal infinite sums:

\[ f(x_1, ..., x_n) = \sum_{k_1, ..., k_n=0}^{\infty} c_{k_1, ..., k_n} x_1^{k_1} ... x_n^{k_n} \]

In this ring, show that:

- prove that $f$ is nilpotent only if all the $c_{k_1, ..., k_n}$ are nilpotent (explain why the “if” statement fails)
- prove that $f$ is a unit if and only if $c_{0, ..., 0}$ is a unit

2) In any ring $R$, an element $e \in R$ is called an idempotent if $e^2 = e$. Given two rings $R_1$ and $R_2$, the product ring $R_1 \times R_2$ is defined with component-wise addition and multiplication (also set $0 = (0, 0)$ and $1 = (1, 1)$).

Fix $R$ commutative. To any idemponent of $R$, associate a decomposition:

\[ R = R_1 \times R_2 \]

and to every such decomposition, associate an idempotent of $R$. 