18.702 Comments on Problem Set 7

The first two problems are relatively easy.

3. Chapter 14, Problem 4.5. (lattices in the plane)

Let $L$ denote the subgroup of $\mathbb{C}$ generated by $\alpha, \beta, \gamma$. If the three numbers lie on a line through 0, they don’t span a lattice. Suppose that they don’t lie on a line. Then $L$ will be a lattice if and only if there is an integer dependence relation $a\alpha + b\beta + c\gamma = 0$ with $a, b, c \in \mathbb{Z}$. The proof is as follows:

Let’s say that $\alpha$ and $\beta$ are independent. Then $\gamma$ will be a real combination $r\alpha + s\beta$. If $r$ and $s$ are rational numbers, and if $d$ is a common denominator for $r$ and $s$, $L$ will be a subgroup of the lattice with basis $\alpha/d, \beta/d$. Then $L$ will be a discrete subgroup that contains the two independent elements $\alpha, \beta$, and therefore $L$ will be a lattice.

Let $M$ be the lattice spanned by $\alpha$ and $\beta$. If $r$ and $s$ aren’t both rational, no integer multiple of $\gamma = r\alpha + s\beta$ will be in $M$. Then, when $m$ and $n$ are distinct integers, $(m-n)\gamma$ won’t be in $M$. The cosets $m\gamma + M$ and $n\gamma + M$ will be distinct, and therefore disjoint. Each coset $m\gamma + M$ contains an element in the parallelogram with vertices 0, $\alpha, \beta, \alpha + \beta$. This gives us infinitely many distinct elements of $L$ in that parallelogram. So $L$ is not discrete, and therefore is not a lattice.

4. Chapter 14, Problem M.5. (matrices that send a lattice to itself)

Let $L_0$ denote the standard lattice $\mathbb{Z}^2$ in $\mathbb{R}^2$. The matrices that stabilize $L_0$ are the invertible integer matrices – the integer matrices with determinant $\pm 1$. You will be able to show this.

If $L$ is any lattice, there will be an invertible real matrix $P$ such that $PL = L_0$. If $A$ is an invertible real matrix $A$ such that $AL = L$, then $PAP^{-1}L_0 = L_0$. Therefore $PAP^{-1}$ is an invertible integer matrix.

This is one answer to the question: The matrices $A$ that stabilize a lattice are those that are conjugate to invertible integer matrices in $GL_2(\mathbb{R})$. However, how can we decide whether or not a given matrix is conjugate to an invertible integer matrix? A better answer is that the matrices that stabilize a lattice are those whose trace is an integer, and whose determinant is $\pm 1$.

If $PAP^{-1}$ is an invertible integer matrix, its characteristic polynomial will have the form $t^2 - at + 1$, where $a = \text{trace}(PAP^{-1})$ is an integer, and this will also be the characteristic polynomial of $A$. (The characteristic polynomials of $A$ and $PAP^{-1}$ are equal.)

Conversely, let $A$ be a real matrix with integer trace and with determinant $\pm 1$. Let $T$ be the linear operator of multiplication by $A$ on $\mathbb{R}^2$. We choose a vector $v_1$ in $\mathbb{R}^2$ that isn’t an eigenvector of $T$. (This is a trick that was used in 18.701 to prove that $PSL_2$ is simple.) Then $v_1$ and $v_2 = Tv_1$ are independent, so $(v_1, v_2)$ is a basis of $\mathbb{R}^2$. We write $Tv_2$ in terms of this basis, $Tv_2 = rv_1 + sv_2$. Then the matrix of $T$ with respect to the basis $(v_1, v_2)$ is

$$B = \begin{pmatrix} 0 & r \\ 1 & s \end{pmatrix}$$

and it a conjugate of $A$. The characteristic polynomial of $B$ is $t^2 - st - r$. So $s = \text{trace}(A)$ and $r = \pm 1$. Therefore $B$ is an invertible integer matrix.

The trick assumes that there is a vector $v_1$ that isn’t an eigenvector of $A$. If every vector is an eigenvector, then $A = cI$. That case is OK too.