Comments on Problem Set 5

1. Chapter 12, Exc. 4.8. (factoring certain quartics)

One case in which \( f = +bx^2 + c \) factors is that the quadratic polynomial \( y^2 + by + c \) has a root in the field \( F \). If \( y^2 + by + c = (y - u)(y - v) \), then, setting \( y = x^2 \), one sees that \( f = (x^2 - u)(x^2 - v) \).

I assigned this problem because there is another way that \( f \) might factor. It can be found by solving the equation \( x^4 + bx^2 + c = (x^2 + px + q)(x^2 + p'x + q') \) with indeterminate coefficients:

\[
(x^2 + px + q)(x^2 + p'x + q') = x^4 + (p + p')x^3 + (q + q' + pp')x^2 + (pq' + p'q)x + qq'
\]

Evaluating coefficients, \( p' = -p \), \( q' = q \), \( q^2 = c \), and \( 2q - p^2 = b \). If these equations can be solved in \( F \), then \( f = (x^2 + px + q)(x^2 - px + q) \).

2. Chapter 12, Exercise 4.5. (irreducibility of some polynomials)

They are all irreducible. Eisenstein applies to (a) and (d), the only possible integer roots of (c) are \( \pm 1 \). One way to analyze (b) is to change variable to \( y = x^{-1} \), obtaining the equation \( y^3 - 6y + 8 \). Then one can check that \( \pm 2^k \) with \( k \leq 3 \) aren’t roots.

3. Chapter 13, Exc. 3.2. (is lattice an ideal?)

To show that a lattice \( L \) with lattice basis \( \alpha, \beta \) is an ideal, it suffices to show that \( \alpha\delta \) and \( \beta\delta \) are in \( L \).

For example, if the lattice basis is \( (5, 1 + \delta) \), we ask: are \( 5\delta \) and \( (1 + \delta)\delta \) in \( L \)? In this case, \( 5\delta = -1 \cdot 5 + 5(1 + \delta) \). However, \( (1 + \delta)\delta = \delta - 5 \) isn’t in the lattice.

4. Chapter 13, Exc. 3.4 (a - d). (shapes of ideals)

You may base reasoning on pictures of the lattices, provided that they are accurate.

(c) The ideals are the principal ideals and the ideals with lattice basis of the form \( \alpha, \frac{1}{2}\alpha\delta \).