Comments on Problem Set 2

1. Chapter 10, Exercise 6.5. (the standard representation of $S_n$)

As is true for any permutation representation, the trivial representation is the summand that corresponds to the invariant subspace $U$ spanned by $(1,\ldots,1)^t$. Since permutation matrices are orthogonal, and therefore unitary, $W = U^\perp$ is an invariant subspace. It is the space of vectors $(a_1,\ldots,a_n)^t$ such that $\sum a_i = 0$, and its dimension is $n - 1$.

Let $w$ be an nonzero vector in $W$. In 18.701, in problem M1 of Chapter 4, you proved that the span of the orbit of $w$ can have dimension 0, 1, $n-1$, or $n$, and that the first two cases occur when the vector is constant: $(a,a,\ldots,a)^t$.

Since $w$ is in $W$, the sum of its entries is zero. So, since $w$ isn’t zero, it isn’t a constant vector. The dimension of its span is $n - 1$. Every nonzero vector $w$ in $W$ spans $W$. Therefore $W$ is irreducible.

3. Chapter 10, Exercise M.9 (Frobenius Reciprocity)

As mentioned in the pset, this comes out when one writes what has to be proved carefully.

(a) To show that $\text{ind } S$ is a representation of $G$, you have to compute the four products $(\text{ind } S)_h(\text{ind } S)_{g'}$, $(\text{ind } S)_{g'}(\text{ind } S)_h$, $(\text{ind } S)_h(\text{ind } S)_{h'}$, and $(\text{ind } S)_{g'}(\text{ind } S)_{g'}$.

The character $\chi_{\text{ind } S}$ of $\text{ind } S$ is zero on the coset $aH$, and $\chi_{\text{ind } S}(h) = \chi_S(h) + \chi_S(a^{-1}ha)$ when $h$ is in $H$.

(c) The definition of $\langle , \rangle$ shows that $\langle \chi_{\text{ind } S}, \chi_R \rangle = \frac{1}{|G|} \left( \sum_h \chi_S(h)\chi_R(h) + \sum_h \chi_S(a^{-1}ha)\chi_R(h) + 0 \right)$

Since $|G| = 2|H|$, the first sum, divided by $|G|$, is $\frac{1}{2} < \chi_S, \chi_{\text{res } R} >$. The only tricky part is the second sum. Since $h$ and $a^{-1}ha$ are conjugate in $G$ and since $R$ is a representation of $G$, $\chi_R(h) = \chi_R(a^{-1}ha)$. So we can replace that sum by $\sum_h \chi_S(a^{-1}ha)\chi_R(a^{-1}ha)$, which is equal to $\sum_h \chi_S(h)\chi_R(h)$, summed in another order.

(d) Looking at the definition of $\text{ind } S$, one sees that $\text{res}(\text{ind } S) = S \oplus S'$. Frobenius Reciprocity tells us that $< \text{ind } S, \text{ind } S > = < S, \text{res}(\text{ind } S) > = < S, S \oplus S' > = < S, S > + < S, S' >$

The right side of this equation is 1 if $S \not\approx S'$ and 2 if $S \approx S'$.
4. **Determine the character table of the symmetric group \( S_5 \).**

Two elements are conjugate in the symmetric group if their cycle decompositions have the same sizes. The cycle decompositions are indicated by dots in the table.

The character table consists of the first seven rows below. The classes to the left of the second vertical line are the even conjugacy classes. Because there are so many cross-checks, there is more than one way to determine the table.

<table>
<thead>
<tr>
<th>size</th>
<th>(1)</th>
<th>(15)</th>
<th>(20)</th>
<th>(24)</th>
<th>(10)</th>
<th>(20)</th>
<th>(30)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \chi_1 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_2 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>( \chi_3 )</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>2</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>( \chi_4 )</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>-2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( \chi_5 )</td>
<td>5</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>( \chi_6 )</td>
<td>5</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_7 )</td>
<td>6</td>
<td>-2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( perm )</td>
<td>5</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(..)</td>
<td>10</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>4</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Notice that the character values are integers. It isn’t hard to verify this fact. A permutation is conjugate to its inverse, so when \( g \) is a 4-cycle, for example, \( i \) and \( -i \) appear as eigenvalues of \( \rho_g \) the same number of times.

**Method 1: Using permutation representations**

In the table, \( \chi_2 \) is the sign character, \( \chi_3 \) is determined by decomposing the permutation representation that corresponds to the operation of \( S_5 \) on the set of five indices. That character is labelled \( perm \). Then \( \chi_4 = \chi_2 \chi_3 \) (as discussed in Problem 2).

Next, we compute the permutation representation for the operation by conjugation on on the conjugacy class (..). Its character is the last row of the table. To compute the value of this character on the 3-cycle \((1 2 3)\), for example, one has to determine the number of transpositions \((i j)\) that commute with a 3-cycle: \((1 2 3)(i j) = (i j)(1 2 3)\). The answer is that the transposition \((4 5)\) is the only one. So the value of the character in that position is 1.

Using the projection formula, one sees that the character labelled (..) is the sum of three irreducible characters, including \( \chi_1 \) and \( \chi_3 \). So \((..) - \chi_1 - \chi_3\) is an irreducible character. It is labelled \( \chi_5 \), and \( \chi_6 = \chi_5 \chi_2 \). Then \( \chi_7 \) is determined by orthogonality.
Method 2: Using Frobenius Reciprocity

We look at the character table (10.6.14) of the alternating group $H = A_5 (= I)$. Referring to part (d) of problem M.9, we see that since there is just one irreducible representation of $H$ of dimension 4, it is isomorphic to its conjugate representation, and therefore $\text{ind} S$ is a sum of two non-isomorphic representations, both having dimension 4. So $G = S_5$ has two irreducible characters of dimension 4. The values of their characters on even permutations are determined from the table for $H$. Similarly, there are two irreducible characters of dimension 5, whose values on even permutations are determined by the table for $H$. This leaves just one irreducible character of dimension 6. It is induced from either of the two characters of $H$ of dimension 3, and its character values are determined for all permutations by the table for $H$ and the definition of the induced character.

This leaves the character values of $\chi_3, 4, 5, 6$ on odd permutations to be determined.

The characters called $\chi_3$ and $\chi_4$ can be interchanged, and $\chi_3$ is the product $\chi_4 \chi_2$.

The possible values for $\chi_4$ in the classes (..) and (....) are 4, 2, 0, -2, -4. Computing $< \chi_4, \chi_4 >$ shows that $\pm 4$ is too big for either class, and that $\pm 2$ is too big for the class (....). Thus $\chi_4(....) = 0$. If $\chi_4(..)$ were zero too, orthogonality with $\chi_1$ would imply $\chi_4(\ldots)(..) = 0$ as well. Then we would have $\chi_3 = \chi_4$, which is impossible. Thus $\chi_4(..) = \pm 2$ and $\chi_4(\ldots)(..) = \mp 1$. Similar reasoning determines $\chi_5$ and $\chi_6$. 

3