18.701 Comments on Quiz 2

You are expected to justify your assertions, but you may state and use without proof results from lectures or from the assigned reading, unless you are asked to prove them here.

1. (15 points) The matrix below represents a rotation of \( \mathbb{R}^3 \). Determine its axis of rotation and its angle \( \pm \theta \) of rotation. (The angle is determined only up to sign.)

\[
\begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

The axis of rotation is spanned by the vector \((1, 1, 1)^t\). The angle is \( \pm 2\pi/3 \). One can determine the angle by noticing that the cube of this matrix is the identity. Or one can use the formula \( \text{trace } A = 2 \cos \theta + 1 \) for a rotation with angle \( \theta \).

2. (15 points) In the group \( M \) of isometries of the plane, let \( f \) be a glide reflection with horizontal glide line, and let \( g \) be a glide reflection with vertical glide line. The composition \( fg \) is a rotation. Determine the possible angles that this rotation might have.

The angle is \( \pi \). The image of \( f \) in the point group is the reflection \( \overline{r} \) with horizontal line of reflection. The image of \( g \) in the point group is the reflection \( \overline{s} \) with vertical line of reflection. The product \( \overline{rs} \) is the rotation \( \overline{\rho} \) with angle \( \pi \). The map \( G \to \overline{G} \) is a homomorphism, so \( fg \) is a rotation with that same angle.

3. (15 points) The figure below is supposed to extend indefinitely in all directions. Let \( G \) be its group of symmetries. Determine the point group of \( G \).

The pattern has reflections about horizontal axes, glide reflections with vertical line of reflection, and rotations by \( \pi \). The point group is the dihedral group \( D_2 = \{ \overline{1}, \overline{p}, \overline{r}, \overline{s} \} \) where \( \overline{p} \) is rotation with angle \( \pi \) and \( \overline{s} = \overline{p} \overline{r} \) is reflection about a vertical axis. This reflection is represented only by glides in the pattern.
4. (15 points) Let $G$ be a group of order $105 = 3 \cdot 5 \cdot 7$. How many elements of order 5 might $G$ contain?

The Sylow Theorems tell us that $G$ contains either 1 or 21 subgroups of order 5. Each subgroup contains 4 elements of order 5, and distinct subgroups of order 5 can intersect only in the identity. So there are either 4 or 84 elements of order 5 in $G$. Full credit given for this.

However, it happens that 84 elements of order 5 isn’t possible. If there were 84 such elements, there would be just 21 elements left over, including the identity. The third Sylow Theorem tells us that the number of Sylow 7 groups is either 1 or 15. There isn’t room for 15 such groups, so the 7-group is normal. Let $x$ be a generator for the Sylow 7 group $N$, and let $y$ be a generator for one of the Sylow 5 groups $H$. Since $N$ is normal, $yxy^{-1} = x^i$ for some $i$, $1 \leq i < 7$. Since $y^5 = 1$, $x = y^5xy^{-1} = x^5$. So $i^5 \equiv 1 \pmod{7}$. The only such index is $i = 1$. Therefore $yxy^{-1} = x$, and $yx = xy$. This shows that the centralizer of $y$ contains $x$ and $y$, so it has order at least 35. Then $H$ can not have 21 conjugate subgroups.

5. (15 points) Let $H$ be a subgroup of a group $G$. The group $G$ operates by left multiplication on the set of left cosets of $H$: A group element $g$ acts on the coset $[aH]$ as $g \ast [aH] = [gaH]$. Determine the stabilizer of the coset $[aH]$.

The stabilizer is the conjugate subgroup $aHa^{-1}$.

6. (25 points) Determine the class equation of the alternating group $A_4$.

The elements of $A_4$ are: the identity, eight 3-cycles, and three products of disjoint transpositions. In the symmetric group the eight 3-cycles are conjugate and so are the three products of transpositions: $|A_4| = 12 = 1 + 8 + 3$.

A conjugacy class in $S_4$ is either a conjugacy class in $A_4$, or else it splits into two classes, each containing a half of the number of elements. The order of a conjugacy class in $A_4$ divides the order 12 of $A_4$. Since eight doesn’t divide 12, the class of 3-cycles splits. Since three is odd, the other class doesn’t split.

The Class Equation is $12 = 1 + 4 + 4 + 3$. 