1. Chapter 7, Exercise 8.6. (groups of order 55)

The Third Sylow Theorem tells us that there is just one subgroup of order 11, so it is a normal subgroup. Also, the number of Sylow 5 subgroups is either 1 or 11. Let $x$ generate the Sylow 11 subgroup, and let $y$ generate one of the Sylow 5 subgroups. Then because $< x >$ is normal, $yxy^{-1} = x^i$ for some $i$, $1 \leq i < 11$. Since $y^5 = 1$, $x = y^5xy^{-5} = x^5$. So $i^5 \equiv 1$ modulo 11. The exponents that have this property are 1, 3, 4, 5, 9. We can change the generator $y$ for the Sylow 5 group to $z = y^2$ without changing the group. The conjugation relation becomes $zxz^{-1} = y^2xy^{-1} = x^{i^2}$. This changes the possible exponents as follows: $3 \rightarrow 9 \rightarrow 4 \rightarrow 5 \rightarrow 3$. So all of the possible exponents different from 1 give isomorphic groups. And of course, $1 \rightarrow 1$.

When we know that $yxy^{-1} = x^i$, we can write this relation as $yx = x^iy$, which tells us how to put a product of $x$'s and $y$'s into the form $x^ry^s$, with $0 \leq r < 11$ and $0 \leq s < 5$. The group is determined. Thus there are at most two isomorphism classes of groups of order 55. The case $yxy^{-1} = x$ is the abelian case, which exists. It is the cyclic group of order 55.

Does the case $yxy^{-1} = x^3$ exist? One way to show that it does is to use Todd-Coxeter. Another way is to find the group. We need an element of order 11. Let’s look in the symmetric group $S_{11}$, using indices 0, ..., 10. Let $x$ be the 11-cycle $(0123\cdots10)$. A little experimentation with 5-cycles shows that $y = (13954)(267108)$ works: $yxy^{-1} = x^3$. So this group also exists.

Note: If we tried to define the group by $yxy^{-1} = x^2$, it would collapse. The three relations $x^{11} = 1$, $y^5 = 1$ and $yxy^{-1} = x^2$ define the cyclic group of order 5.

2. Use the Todd-Coxeter Algorithm to determine the order of the group generated by two elements $x, y$,

(a) with relations $x^3 = 1$, $y^2 = 1$, and $yxyxy = 1$.

This is the trivial group.

(b) with relations $x^3 = 1$, $y^4 = 1$, and $xyxy = 1$.

This is a group of order 24. It happens to be the octahedral group of rotational symmetries of an octahedron or a cube. One can take for $y$ rotation by $\pi/2$ about a face and for $x$ rotation by $2\pi/3$ about a vertex, both counterclockwise.
3. Chapter 7, Exercise M.1. Classify groups generated by two elements \( x, y \) of order two.

The element \( z = xy \) is useful for this.

The given relations in \( G \) are \( x^2 = 1 \) and \( y^2 = 1 \). When we use \( z = xy \), we can eliminate one of the generators, say \( y = xz \). Then the relation \( y^2 = 1 \) becomes \( xz = 1 \), which can be written as \( z = xz^{-1} \). This allows us to write any element \( g \) of \( G \) as a product \( g = x^i z^j \), where \( i = 0, 1 \) and \( j \) is an integer.

If there are no other relations between \( x, z \), the expression for \( g \) is unique. The group is infinite. It is called the infinite dihedral group.

If there is another relation, it will be either \( z^k = 1 \) or \( xz^k = 1 \). When \( z^k = 1 \), we can assume that \( k \) is positive. The relations \( x^2 = 1, z^k = 1, xz = 1 \) define the dihedral group \( D_k \). When \( xz^k = 1 \), we get \( 1 = xz^k xz^k = x(x^{-1} z^k)z^k = z^{2k} \). The relations \( x^2 = 1, z^{2k} = 1, xz = 1 \) define the dihedral group \( D_{2k} \).

All that remains is to describe the possible quotient groups of a dihedral group \( D_k \) that can be obtained by introducing more relations. Adding a relation \( z^j = 1 \) simply reduces the integer \( k \) to the g.c.d. \( \delta \) of \( k \) and \( j \). The group becomes the dihedral group \( D_{\delta} \).

Suppose we add a relation \( xz^j = 1 \). Let \( w = xz^j \). Since \( zz = xz^{-1}, w^2 = xz^j xz^{-j} = x(xz^{-j})z^j = 1 \), and \( zw = wz^{-1} \). When we replace the generator \( x \) by \( w \), we obtain the relations \( w^2 = 1, z^k = 1, zw = wz^{-1} \), another representation of the dihedral group \( D_k \). Now setting \( w = 1 \), the quotient group is generated by \( z \) with the relations \( z^k = 1 \) and \( z^2 = 1 \). It is either the trivial group or the cyclic group of order 2.

Since \( x \) and \( y \) were supposed to be elements of order 2, the trivial group is ruled out.