1. Chapter 6, Exercise M.4. (*hypercube*)

Let $G_n$ be the group of orthogonal operators $M$ that are symmetries of the hypercube $C_n$, represented by orthogonal matrices. Let’s call “signed permutations” the matrices that are obtained from permutation matrices by changing some of the entries 1 to $-1$. The signed permutations form a subgroup $H_n$ of $G_n$, of order $2^n \cdot n!$. We’ll show by induction that $H_n = G_n$.

Let $F$ be the face hypercube of dimension $n - 1$ of points of $C_n$ at which $x_n = 1$. The elements of $G_n$ that stabilize $F$ (that send $F$ to $F$) are those that fix the last coordinate of a vector. They have block form $M = \begin{pmatrix} A & 0 \\ C & 1 \end{pmatrix}$, where $A$ is an $(n-1) \times (n-1)$ matrix and $C$ is an $(n-1)$-dimensional row vector. Since $M$ is orthogonal, $C = 0$, and $A$ is an orthogonal matrix. Since $M$ is a symmetry of the hypercube $C_n$ that sends $F$ to $F$, it defines a symmetry of $F$. Therefore $A$ is an element of $G_{n-1}$. By induction, $H_{n-1} \approx G_{n-1}$. So the stabilizer of $F$ has order $2^{n-1} \cdot (n-1)!$.

There are $n$ faces defined by $x_i = 1$ and $n$ faces with $x_i = -1$. These $2n$ faces form a $G_n$-orbit. The counting formula shows that $|G_n| = 2n|G_{n-1}| = 2^n \cdot n! = |H_n|$. Thus $H_n = G_n$.

The dihedral group of symmetries of the square is represented here in an interesting way, as the group whose elements are the eight signed permutation matrices $\begin{pmatrix} \pm 1 & \pm 1 \\ \pm 1 & \pm 1 \end{pmatrix}$.
2. **Determine the Class Equation of the group** $G = GL_3(\mathbb{F}_2)$ of invertible $3 \times 3$ matrices with entries mod 2.

The order of $G$ is 168.

The characteristic polynomial of an element $A$ of $G$ can be any cubic polynomial whose constant term $\det A$ is $-1$, and $-1 = +1$. There are four such polynomials: $t^3 + t^2 + t + 1$, $t^3 + 1$, $t^3 + t + 1$, $t^3 + t^2 + 1$.

**Aside:** In case you had to work to compute the coefficient of $t$ in the characteristic polynomial. The coefficient of $t$ in the characteristic polynomial of the $3 \times 3$ matrix $(a_{ij})$, is the sum of the three symmetric $2 \times 2$ minors:

$$
\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \det \begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix} + \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}
$$

The first of the four polynomials listed above has $t = 1$ as a triple root: $(t + 1)^3 = t^3 + t^2 + t + 1$ modulo 2. There are three classes of matrices with this characteristic polynomial, including the identity. The other classes are represented by the matrices

$$
A_1 = \begin{pmatrix} 1 & 1 & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 1 & 1 \\ \cdot & 1 \\ \cdot & 1 \end{pmatrix}.
$$

The centralizer of $A_1$ is the same as that of the simpler matrix $e_{12} = I + A_1$. We compute the centralizer of $e_{12}$ by solving the equation $P e_{12} = e_{12} P$ with indeterminate $P$:

$$
\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} = \begin{pmatrix} a & \cdot & \cdot \\ \cdot & d & \cdot \\ \cdot & \cdot & g \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} \begin{pmatrix} 1 & b & c \\ a & \cdot & \cdot \\ \cdot & h & i \end{pmatrix} = \begin{pmatrix} d & e & f \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}
$$

If $P$ commutes with $e_{12}$, $d = f = g = 0$ and $a = c$. So $P$ has the form

$$
\begin{pmatrix} a & b & c \\ \cdot & a & \cdot \\ \cdot & h & i \end{pmatrix}
$$

Since $P$ must be invertible, $a = i = 1$, while $b, c, h$ can be arbitrary elements of $\mathbb{F}_2$. The centralizer $Z(e_{12}) = Z(A_1 n)$ has order 8, and the conjugacy class $C(A_1)$ has order $168/8 = 21$.

A similar computation for the matrix $A_2$ shows that its centralizer consists of invertible matrices of the form

$$
P = \begin{pmatrix} a & b & c \\ \cdot & a & b \\ \cdot & \cdot & a \end{pmatrix}
$$

Here $a$ must be 1, while $b$ and $c$ can be arbitrary. The centralizer of $A_2$ has order 4 and the conjugacy class $C(A_2)$ has order $268/4 = 42$.

The cyclic permutation matrix

$$
B = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}
$$

is a good choice for a matrix with characteristic polynomial $t^4 + 1$. Its centralizer consists of the three matrices $I, B, B^2$. So $Z(B)$ has order 3 and $C(B)$ has order 56.
The matrices for the two remaining characteristic polynomials aren’t quite so simple. One matrix with characteristic polynomial \( t^3 + t^2 + 1 \) is
\[
D = \begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{pmatrix}
\]
and \( E = I + D \) has characteristic polynomial \( t^3 + t + 1 \). The centralizers of \( D \) and \( E \) are the same, so their conjugacy classes have the same order. We compute the centralizer of \( D \):
\[
\begin{pmatrix}
a & b & c \\
d & e & f \\
g & h & i \\
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{pmatrix}
= \begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
a & b & c \\
an+b & a+c & b \\
d+e & d+f & e \\
g+h & g+i & h \\
\end{pmatrix}
= \begin{pmatrix}
a & b & c \\
(a+d) & (b+e) & (c+f) \\
(a+g) & (b+h) & (c+i) \\
\end{pmatrix}
\]
Equating these matrices, one finds that an element in the centralizer must have the form
\[
P = \begin{pmatrix}
a & b & c \\
b & a+b+c & b+c \\
c & b+c & a+b \\
\end{pmatrix}
\]
The determinant is
\[
\det P = a^3 + b^3 + c^3 + ab^2 + bc^2 + ca^2 + abc
\]
There are 8 such matrices, and all of them except 0 have determinant 1. The centralizer of \( D \) consists of the 7 nonzero matrices. They happen to be the powers of \( D \):
\[
D^1 = \begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{pmatrix}
, \\
D^2 = \begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{pmatrix}
, \\
D^3 = \begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{pmatrix}
, \\
D^4 = \begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{pmatrix}
, \\
D^5 = \begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{pmatrix}
= E \\
D^6 = \begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{pmatrix}
, \\
D^7 = I
\]
So the Class Equation of \( GL_3(\mathbb{F}_2) \) is
\[
168 = 1 + 21 + 42 + 56 + 24 + 24
\]
Note that there are rather few conjugacy classes. One can use the Class Equation to show that \( GL_2(\mathbb{F}_2) \) is a simple group, by verifying that the only sums of elements on the right side of the equation that includes 1 and that divides 168 are the trivial sums: 1 and 168. The group is the second smallest simple group, apart from the groups of prime order. The smallest nonabelian simple group is \( I \approx A_5 \).
3. Chapter 7, Exercise 5.12. (class equations of \(S_6\) and \(A_6\))

The conjugacy classes in \(S_6\) correspond to the partitions of 6, which are listed below. Because we know the conjugacy classes, this is one case in which computing the orders of the conjugacy classes is simpler than computing their centralizers, though it is easy to make a mistake.

\[
\begin{align*}
C_1 & : 1 + 1 + 1 + 1 + 1 + 1 \\
C_2 & : 2 + 1 + 1 + 1 + 1 \\
C_3 & : 2 + 2 + 1 + 1 \\
C_4 & : 2 + 2 + 2 \\
C_5 & : 3 + 1 + 1 + 1 \\
C_6 & : 3 + 2 + 1 \\
C_7 & : 3 + 3 \\
C_8 & : 4 + 1 + 1 \\
C_9 & : 4 + 2 \\
C_{10} & : 5 + 1 \\
C_{11} & : 6
\end{align*}
\]

The orders of these conjugacy classes:

\[
\begin{align*}
|C_1| & = 1, \ |C_2| = 15, \ |C_3| = 45, \ |C_4| = 15, \ |C_5| = 40, \ |C_6| = 120, \\
|C_7| & = 40, \ |C_8| = 90, \ |C_9| = 90, \ |C_{10}| = 144, \ |C_{11}| = 120
\end{align*}
\]

So the Class Equation of \(S_6\) is

\[
720 = 1 + 15 + 45 + 15 + 40 + 120 + 40 + 90 + 90 + 144 + 120
\]

The conjugacy classes consisting of even permutations are \(C_1\), \(C_3\), \(C_5\), \(C_7\), \(C_9\), and \(C_{10}\). They have orders 1, 45, 40, 90, 90, and 144, respectively. We are supposed to know that, if \(C_S(p)\) is the conjugacy class in \(S_6\) of an even permutation \(p\) there are two cases: If the centralizer \(Z_S(p)\) of \(p\) in \(S_6\) contains an odd permutation, \(C_S(p)\) is also the conjugacy class \(C_A(p)\) of \(p\) in \(A_6\). If the centralizer \(Z_S(p)\) contains only even permutations, then the class \(C_S(p)\) splits into two conjugacy classes in \(A_6\), each having a half of the order of \(C_S(p)\). The first two classes don't split because they have odd order. The conjugacy classes \(C_5\), \(C_7\), \(C_9\) don't split either. The only tricky case is \(C_7\). One has to notice that the permutation \(p = (1\ 2\ 3)(4\ 5\ 6)\) commutes with the odd permutation \(q = (1\ 4)(2\ 5)(3\ 6)\). The last class splits because 144 doesn't divide 360.

The Class Equation of \(A_6\) is

\[
360 = 1 + 45 + 40 + 40 + 90 + 72 + 72
\]

Looking at this equation, one sees that \(A_6\) is a simple group.