1. Chapter 4, Exercise M.7a,b (*powers of an operator*

(b) The conditions are equivalent. To show this, it is essential to write down carefully what the conditions (3) and (4) mean.

By (1), we know that \( W_{r+1} \subset W_r \) and that \( K_r \subset K_{r+1} \).

(1) \( \Leftrightarrow \) (3): Condition (3): \( W_r \cap K_1 = \{ 0 \} \) can be stated this way: If \( w \in W_r \), then \( T(w) \neq 0 \).

Suppose that (3) is true, and let \( x \) be an element of \( K_{r+1} \), and let \( w = T^r(x) \). So \( w \) is in \( W_r \), and \( T(w) = T^{r+1}(x) = 0 \). So \( w = 0 \), and this shows that \( x \) is in \( K_r \). Therefore \( K_{r+1} \subset K_r \), and \( K_{r+1} = K_r \). So (3) \( \Rightarrow \) (1).

Conversely, suppose that (1) holds and let \( w \) be a nonzero element of \( W_r \). Then \( w = T^r(x) \) for some \( x \) in \( K_r \). So \( x \notin K_{r+1} \), and \( w \notin K_1 \). Therefore \( W_r \cap K_1 = \{ 0 \} \), and (1) \( \Rightarrow \) (3).

(2) \( \Leftrightarrow \) (4):

We write condition (4) this way: Any \( v \in V \) can be written as a sum \( v = w + u \) with \( w \in W_1 \) and \( u \in K_r \). Then \( w = T(x) \) for some \( x \), and \( T^r(u) = 0 \). So \( T^r(v) = T^r(w) + 0 = T^{r+1}(x) \). This tells us that \( W_r \subset W_{r+1} \). So \( W_r = W_{r+1} \). Therefore (4) \( \Rightarrow \) (2).

Conversely, suppose (2) holds, and let \( v \in V \). Then \( T^r(v) = T^{r+1}(x) \) for some \( x \). Let \( w = T(x) \) and \( u = v - w \). Then \( T^r(u) = 0 \), so \( u \in K_r \). Since \( v = w + u \), this shows that \( W_1 + K_r = V \). Therefore (2) \( \Rightarrow \) (4).

When \( V \) has finite dimension, the dimension formula \( \dim V = \dim K_r + \dim W_r \) shows that (1) \( \Leftrightarrow \) (2).

When the dimension of \( V \) is infinite, this is no longer true, as is shown by the shift operators on \( V = \mathbb{R}^\infty \).

The right shift sends \((a_1,a_2,...)\) to \((0,a_1,a_2,...)\). For this operator, \( K_r = 0 \) for all \( r \) and \( W_r \) is strictly descending. Then (1),(3) are true for all \( r \), and (2),(4) are false for all \( r \).

The left shift sends \((a_1,a_2,...)\) to \((a_2,a_3,...)\). For this operator, \( K_r \) is strictly increasing and \( W_r = V \) for all \( r \). Then (1),(3) are false for all \( r \), and (2),(4) are true for all \( r \).

2. Chapter 5, Exercise 1.5. (*fixed vector of a rotation matrix*)

Let \( A \) be a rotation matrix, an element of \( S_3 \). If a vector \( X \) is fixed by \( A \), it is also fixed by its inverse \( A^t \), and therefore \( MX = (A - A^t)X = 0 \). The rank of \( M \) is less than 3. Conversely, if \( MX = 0 \), then \( AX = A^{-1}X \). When the angle of rotation isn’t 0 or \( \pi \), this happens only for vectors \( X \) in the axis of rotation, so the rank of \( M \) is 2.

A fixed vector can be found by solving the equation \( MX = 0 \). Let \( u = a_{12} - a_{21} \), \( v = a_{13} - a_{31} \), \( w = a_{23} - a_{32} \). Then

\[
M = \begin{pmatrix} 0 & u & v \\ -u & 0 & w \\ -v & -w & 0 \end{pmatrix}
\]

and \((w,-v,u)^t\) is a fixed vector.
3. Chapter 5, Exercise M.6. (an integral operator)

I like this problem for several reasons. One can’t use the characteristic polynomial, the eigenvalues are unusual, and it has applications.

Suppose that $A = u + v$. Then $A \cdot f = u \int_0^1 f(v)dv + \int_0^1 vf(v)dv = cu + d$, where $c = \int_0^1 f(v)dv$ and $d = \int_0^1 vf(v)dv$. So $A \cdot f$ is always a linear function. Evaluating at two special functions such as $f(u) = 1$ and $f(u) = u$ gives independent linear functions, so the image is the space of all linear functions.

To find eigenvectors with eigenvalues $\lambda \neq 0$, one uses the fact that the image of any function is linear. Therefore an eigenvector must be a linear function. One substitutes a linear function $f = au + b$ with undetermined coefficients and an indeterminate $\lambda$ into the equation $A \cdot f = \lambda f$. This give two equations in the three unknowns $a, b, \lambda$. One can solve because the eigenvector will be determined only up to scalar factor.

4. Chapter 6, Exercise 5.10. (groups containing two rotations)

Let $f$ and $g$ be the two rotations. The elements that one can obtain from them are products of the four elements $f, g, f^{-1}, g^{-1}$. We are looking for a product that is a translation. The simplest way to analyze the situation is to use the homomorphism $M \xrightarrow{\pi} O_2$ from the group $M$ of isometries to the orthogonal group. This homomorphism drops the translation from a product $t_a \rho_\theta$, and keeps the rotation, sending that element to $\rho_\theta$. The kernel of $\pi$ is the group of translations. If $\alpha, \beta$, are the angles of rotation about various points of some isometries $f, g$, then

$$\pi(fg) = \rho_\alpha \rho_\beta = \rho_{\alpha + \beta}.$$  

The angles add. A product will be a translation if and only if it is in the kernel of $\pi$, which happens when the sum of the angles is zero. This being so, we try the commutator $fgf^{-1}g^{-1}$. The sum of the angles is zero, so this is a translation.

However, we need to check that $fgf^{-1}g^{-1}$ isn’t translation by the zero vector. Let’s choose the origin at one of the rotation points, say the rotation $f$. Then $f = \rho_\alpha$, while $g = t_b \rho_\beta$ for some $b$. Then

$$fgf^{-1}g^{-1} = (\rho_\alpha)(t_b \rho_\beta)(\rho^{-\alpha})(\rho^{-\beta}t^{-b}) = \rho_\alpha t_b \rho^{-\alpha} t^{-b} = t_{\rho^{-\alpha}(b)} - b$$