This pset is due Wednesday, October 11.

1. Chapter 3, Exercise M.3. (polynomial paths)

This exercise is designed to teach two simple things:

(a) A set $(v_1, \ldots, v_n)$ of elements of a vector space $V$ of dimension $d$ is dependent if $n > d$.

(b) Two functions $u, v$ satisfy a polynomial equation of degree $\leq d$ if and only if the monomials $u^i v^j$ of degree at most $d$ are dependent. (The degree of $u^i v^j$ is $i+j$.)

(b) Let $u = x(t)$ and $v = y(t)$ be polynomials. The problem is to show that there is a polynomial $F(x, y)$ in variables $x, y$ such that $F(x(t), y(t))$ is the zero polynomial.

Let’s say that $u$ and $v$ have degree $k$ or less. The degree of $F$ isn’t given. It can be an arbitrary integer $d$. The space $V$ of polynomials $F$ of degree at most $d$ has as basis the monomials $x^i y^j$ with $i + j \leq d$. There are $N = (d + 2)(d + 1)/2$ such monomials. When we substitute $u$ and $v$ these monomials, we obtain $N$ polynomials $u^i v^j$ in $t$, of degree at most $kd$. The polynomials of degree at most $kd$ in one variable $t$ form a vector space $V$ of dimension $r = kd + 1$. Since $N$ is quadratic in $d$ while $r$ is linear in $d$, we will have $N > r$ if $d$ is large enough. Then the $N$ polynomials $u^i v^j$ must be dependent.

2. Chapter 4, Exercise 1.5. (about the dimension formula)

(c) I hope that you found the formula $\dim U + \dim V = \dim (U \cap V) + \dim (U + V)$.

3. Chapter 4, Exercise 2.5 (independent rows and columns of a matrix)

Let’s permute rows and columns to make $M$ into the upper left $r \times r$ submatrix of $A$. We can make row operations using the $r$ rows at the top to clear out the rows with indices $> r$. The $r$ rows of $A$ at the top and the $r$ columns on the left of $A$ remain independent. Then we use column operations to clear out the columns with indices $> r$. etc...

4. Chapter 4, Exercise M.1 (permuting entries of a vector)

The answer is that the rank of such a matrix can be $0, 1, n - 1$, or $n$.

Here is one way to show that no other ranks occur: Suppose that the entries $a_i$ aren’t all equal. We can permute the entries so that $a_1 \neq a_2$. Let $\tau$ denote the transposition $(12)$. Let $w = v - \tau v$ Then $w_1 = a_1 - a_2$, $w_2 = a_2 - a_1$, and all other entries of $w$ are zero. The space spanned by the rows of $M$ contains $w$, so it contains $u = (1, -1, 0, \ldots, 0)$, and then it also contains all permutations of $u$. The space spanned by the permutations of $u$ has dimension $n - 1$. It is the space of vectors $v$ such that $v_1 + \cdots + v_n = 0$.

5. Determine the finite-dimensional spaces $W$ of differentiable functions $f(x)$ with this property:

If $f$ is in $W$, then $\frac{df}{dx}$ is in $W$.

The point here is that, if $f$ is a function in $V$, then all of its derivatives will be in $V$. Since $V$ is finite dimensional, the derivatives can’t be independent. There will be some linear relation among them. This means that $f$ solves a homogeneous, constant coefficient, differential equation. Then $f$ is a combination of functions of the form $x^m e^{ax}$ (where $a$ may be complex). Once one has seen this, it isn’t hard to figure out what the finite dimensional spaces are. They will be the span of finitely many such functions $x^m e^{ax}$, the only additional condition being that if $x^m e^{ax}$ is among them, so is $x^{m-1} e^{ax}$.

The space spanned by the functions $e^x, xe^x, x^2e^x, e^{2x}, xe^{2x}$ is a typical example.