18.701 Problem Set 3

due Wednesday, September 26

1. Chapter 3, Exercise 4.4 (order of $GL_2(F_p)$)

2. (a homomorphism from $GL_2(F_3)$ to $S_4$) Let $GL$ denote the group $GL_2(F_3)$ of invertible matrices with entries modulo 3. This group operates on 2-dimensional vectors with entries mod 3 by matrix multiplication, as usual:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

There are 9 vectors modulo 3, and four pairs $\pm v$ of nonzero vectors, namely

$$s_1 = \pm \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ s_2 = \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ s_3 = \pm \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ s_4 = \pm \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

These pairs are permuted by the elements of $GL$. Sending a matrix to the permutation it defines gives us a homomorphism $\varphi$ from $GL$ to the symmetric group $S_4$ of permutations of $\{s_1, s_2, s_3, s_4\}$. For example, if $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\varphi(A)$ is the 3-cycle $(s_2 s_3 s_4)$.

(a) Show that $\varphi$ is a surjective map, and determine its kernel.

(b) Determine the subgroup of $GL$ that corresponds, by the Correspondence Theorem, to the alternating subgroup $A_4$ of $S_4$.

(c) Determine the subgroup of $S_4$ that corresponds to the subgroup of $GL$ of upper triangular matrices.

3. Chapter 2, Exercise M.6a,b (paths in $\mathbb{R}^k$)

4. Chapter 2, Exercise M.7 (paths in $GL_n$)

5. Chapter 2, Exercise M.8a ($SL_n$ is connected)