An Isometry that Fixes the Origin is a Linear Operator

This version proof was found by Evangelos Taratoris. It is simpler than the one by Sharon Hollander that is in the text.

Let \( f \) be an isometry of \( \mathbb{R}^n \) such that \( f(0) = 0 \). As in the text, we use prime notation, writing \( x' \) for \( f(x) \).

Let’s suppose we have verified that \( f \) preserves dot products: \((f(u) \cdot f(v)) = (u \cdot v)\), or

\[(u' \cdot v') = (u \cdot v).\]

See the text for this.

To show that \( f \) is a linear operator, we must show that

\[f(x + y) = f(x) + f(y),\]

and that \( f(cx) = cf(x) \),

for all \( x, y \) and all scalars \( c \). We write \( z = x + y \). Then with the prime notation, the first equality to be shown becomes

\[z' = x' + y'.\]

We prove this by showing that the dot product

\[((z' - x' - y') \cdot (z' - x' - y'))\]

is zero, and that therefore the length of the vector \( z' - x' - y' \) is zero.

We expand this dot product:

\[(*) \quad ((z' - x' - y') \cdot (z' - x' - y')) = (z' \cdot z') + (x' \cdot x') + (y' \cdot y') - 2(z' \cdot x') - 2(z' \cdot y') + 2(x' \cdot y')\]

and compare the expansion to the dot product

\[(**) \quad ((z - x - y) \cdot (z - x - y)) = (z \cdot z) + (x \cdot x) + (y \cdot y) - 2(z \cdot x) - 2(z \cdot y) + 2(x \cdot y)\]

Since \( f \) preserves dot products, the dot products on the right sides of the two equations are equal. The left side of \((**)\) is \((z - x - y) \cdot (z - x - y) = (0 \cdot 0) = 0\). Therefore the left side of \((*)\) is zero too.

The proof of the condition \( f(cx) = cf(x) \) is similar. \(\square\)