An Isometry that Fixes the Origin is a Linear Operator

This proof was found by Evangelos Taratoris. It is simpler than the one by Sharon Hollander that is in the text. Both were students in the class.

Let $f$ be an isometry of $\mathbb{R}^n$ such that $f(0) = 0$. As in the text, we use prime notation, writing $x'$ for $f(x)$.

Let’s suppose we have verified that $f$ preserves dot products: $(f(u) \cdot f(v)) = (u \cdot v)$, or

$$(u' \cdot v') = (u \cdot v).$$

See the text for this.

To show that $f$ is a linear operator, we must show that

$$f(x + y) = f(x) + f(y),$$

and that $f(cx) = cf(x)$,

for all $x, y$ and all scalars $c$. We write $z = x + y$. Then with the prime notation, the first equality to be shown becomes

$$z' = x' + y'.$$

We prove this by showing that the dot product

$$(z' - x' - y') \cdot (z' - x' - y')$$

is zero, and that therefore the length of $z' - x' - y'$ is zero.

We expand this dot product:

$$(z' - x' - y') \cdot (z' - x' - y') = (z' \cdot z') + (x' \cdot x') + (y' \cdot y') - 2(z' \cdot x') - 2(z' \cdot y') + 2(x' \cdot y')$$

and compare the expansion to the dot product

$$(z - x - y) \cdot (z - x - y) = (z \cdot z) + (x \cdot x) + (y \cdot y) - 2(z \cdot x) - 2(z \cdot y) + 2(x \cdot y)$$

Since $f$ preserves dot products, the dot products on the right sides of the two equations are equal. The left side of (***) is $((z - x - y) \cdot (z - x - y)) = (0 \cdot 0) = 0$. Therefore the left side of (*) is zero too.

The proof of the condition $f(cx) = cf(x)$ is similar. □