Plane Crystallographic Groups with Point Group $D_2$

We describe the possibilities for a discrete group $G$ of isometries of the plane whose translation group $L$ is a lattice and whose point group $\overline{G}$ is the dihedral group $D_2$.

For reference:
- When coordinates are chosen, every isometry can be written as $m = t_v \varphi$, where $\varphi$ is an orthogonal linear operator and $t_v$ is a translation.
- The homomorphism $M \rightarrow O_2$ sends $t_v \varphi$ to $\varphi$. Its kernel is the subgroup of translations in $M$.
- The point group $\overline{G}$ is the image of $G$ in $O_2$ so $\pi$ defines a surjective homomorphism $G \rightarrow \overline{G}$ whose kernel is the group of translations in $G$.
- Let’s denote the group of translations in $G$ by $T$, and the translation group, the additive group of vectors $v$ such that $t_v$ is in $G$, by $L$. Thus $t_v \in T$ if and only if $v \in L$. The translation group $L$ is a lattice if it contains two independent vectors.
- The elements of $\overline{G}$ carry $L$ to $L$.

With suitable coordinates, $\overline{G} = \{I, \rho, s, \rho s\}$, where $\rho$ denotes reflection about the horizontal axis, $s$ denotes reflection about the vertical axis, and $\rho$ denotes rotation through the angle $\pi$ about the origin.

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \rho = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \pi = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

The bars over the letters are there to distinguish elements of $\overline{G}$ from those of $G$. They have no other meaning.

1. Description of the lattice $L$.

Let $u$ be a point of $L$ that isn’t on either coordinate axis. Then $L$ contains the horizontal vector $u + \rho u$ as well as the vertical vector $u + \pi u$. So $L$ contains nonzero horizontal and vertical vectors. We choose a horizontal vector $a = (a_1, 0)^t$ in $L$ of minimal positive length. This can be done because $L$ is a discrete subgroup of $\mathbb{R}^2$. Then the horizontal vectors in $L$ are the integer multiples of $a$. Similarly, we choose a vertical vector $b = (0, b_2)^t$ in $L$ of minimal positive length. The vertical vectors in $L$ are the integer multiples of $b$. Let $L_1$ denote the lattice $a\mathbb{Z} + b\mathbb{Z}$. Also, let $c = \frac{1}{2}(a + b)$ and let $L_2 = a\mathbb{Z} + c\mathbb{Z}$.

**Lemma 1.** Any vector $v$ in $\mathbb{R}^2$, that isn’t in $L_1$, can be written uniquely in the form $v = w + u$, where $w$ is in $L_1$ and $u$ is in the rectangle whose vertices are $0, a, b, a + b$, and not on the ‘far edges’ $[a, a + b]$, or $[b, a + b]$. If $v$ is in $L$, then $u$ is in the interior of the rectangle.

**Proof.** Since $a, b$ are independent, they form a basis of $\mathbb{R}^2$. So $v = xa + yb$ for some $x, y$ in $\mathbb{R}$. We can write $x = m + p$ with $m \in \mathbb{Z}$ and $0 \leq p < 1$, and $y = n + q$ with $n \in \mathbb{Z}$ and $0 \leq q < 1$. Then $w = ma + nb$ is in $L_1$ and $u = pa + qb$ is in the rectangle, not on the far edges. If $v$ is in $L$, then $v$ can’t be on the near edges of the rectangle either, so it is in interior.

**Lemma 2.** $L$ is either $L_1$ or $L_2$.

**Proof.** We note that $b = 2c - a$ is in $L_2$, and therefore $L_1 \subset L_2$. Since $a$ and $b$ are in $L$, $L_1 \subset L$.

Suppose that $L$ contains an element $v$ not in $L_1$. We write $v = w + u$ as in the previous lemma, with $u = (u_1, u_2)^t$ in the interior of the rectangle $0, a, b, a + b$. So $0 < u_1 < a_1$ and $0 < u_2 < b_2$. Since $\overline{G}$ operates on $L$, $u + \rho u = (2u_1, 0)^t$ is in $L$, and since it is horizontal, $u + \rho u$ is an integer multiple of $a$. But $0 < 2u_1 < 2a_1$. The only possibility is that $u_1 = \frac{1}{2}a_1$. Similarly, $u + \pi u = (0, u_2)^t$ is in $L$, and $u_2 = \frac{1}{2}b_2$. So $u = \frac{1}{2}(a + b) = c$. One finds that $L = L_2$.  

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The reflections and glides in \( G \).

We ask: Are the reflections \( \overline{r} \) and \( \overline{s} \) of \( \overline{G} \) the images of reflections in \( G \)? If so, we can put the origin at the intersection of the lines of reflection. Then \( r \) and \( s \) will be in \( G \), and we will be happy.

**Lemma 3.** Let \( v = (v_1, v_2)^t \) be a vector. The isometry \( g = t_v r \) is either a reflection or a glide, and the horizontal line \( \ell : \{ x_2 = \frac{1}{2} v_2 \} \) is the line of reflection or the glide line. Moreover, \( g \) is a reflection about \( \ell \) if and only if \( v \) is vertical: \( v = (0, v_2)^t \).

**proof.** Since \( g \) reverses orientation, it is either a reflection or a glide. It suffices to show that \( g \) carries the line \( \ell \) to itself. The next computation shows this. Let \( x = (x_1, \frac{1}{2} v_2)^t \) be a point of the line \( \ell \).

\[
g(x) = t_v r(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ \frac{1}{2} v_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} x_1 + v_1 \\ \frac{1}{2} v_2 \end{pmatrix}
\]

Since \( \overline{r} \) is in the point group, \( G \) must contain an element \( g = t_v r \) that maps to \( \overline{r} \), though we don’t know whether or not the translation \( t_v \) by itself is an element of \( G \).

We can multiply \( g \) on the left by any element \( t_w \) of \( T \). The result \( t_{w+v} r \) will be another element that maps to \( \overline{r} \). We write \( v = w + u \) as in Lemma 1. Then \( t_{w+v} r = t_u r t_w r \) is an element of \( G \) that maps to \( \overline{r} \), and \( u = pa + qb = (pa_1, qb_2)^t \) with \( 0 \leq p, q < 1 \). We relabel \( t_u r \) as \( g \).

The element \( g^2 = t_{u} r t_{w} r = t_{w+u} r r = t_{u+v} r \) is in \( G \), and therefore \( u + r u = (2pa_1, 0)^t \) is in \( L \). It is an integer multiple of \( a \). Since \( 0 \leq p < 1 \), \( 2pa_1 \) is either 0 or \( a_1 \), and then \( u_1 \) will be 0 or \( \frac{1}{2} a_1 \).

**Lemma 4.** With notation as above,

(i) If \( u_1 = 0 \), then \( u \) is vertical and \( t_u r \) is a reflection. If \( u_1 = \frac{1}{2} a_1 \), then \( t_u r \) is a glide with horizontal glide vector \( \frac{1}{2} a \).

(ii) If \( L = L_2 \), then \( G \) contains reflections that map to \( \overline{r} \) and \( \overline{s} \) in \( \overline{G} \).

**proof.** (ii) Suppose that \( u_1 = \frac{1}{2} a_1 \) and that \( L = L_2 \). Then \( c = \frac{1}{2}(a + b) \) is in \( L \). We multiply \( t_u r \) on the left by \( t_{-c} \), obtaining \( t_{v} r \) where \( v \) is the vertical vector \((0, v_2 - \frac{1}{2} b_2)^t \). Thus \( t_{v} r \) is a reflection. We can apply the analogous reasoning to the element \( \overline{s} \). So if \( L = L_2 \), then \( \overline{r} \) and \( \overline{s} \) are represented by reflections in \( G \).

When \( L = L_1 \) there are four possibilities: Each of the elements \( \overline{r} \) and \( \overline{s} \) will be represented by a reflection or by a glide with glide vector \( \frac{1}{2} a \) or \( \frac{1}{2} b \), respectively.

We may choose coordinates so that the lines of reflection or the glide lines of the elements that represent \( \overline{r} \) and \( \overline{s} \) are the coordinate axes. Then \( \overline{r} \) is represented either by the reflection \( r \) or by the glide \( g_r = t_{\frac{1}{2} a} r \) and \( \overline{s} \) is represented by \( s \) or by the glide \( g_s = t_{\frac{1}{2} b} s \). Moreover, \( G \) cannot contain both \( r \) and \( g_r \) because \( t_{\frac{1}{2} a} \) isn’t in \( T \).

Thus there are four possibilities: \( G \) contains just one of the sets \( S_1 = \{ r, s \} \), \( S_2 = \{ g_r, s \} \), \( S_3 = \{ r, g_s \} \), or \( S_4 = \{ g_r, g_s \} \).

The cases \( S_2 \) and \( S_3 \) can be interchanged by switching the \( x \) and \( y \) coordinates, so they are redundant. We are left with three possibilities for \( G \), when \( L = L_1 \) and one possibility when \( L = L_2 \).

This is confirmed by Table (6.6.2). There are four patterns with point group \( D_2 \), beginning with the pattern of lozenges, the second brick pattern is the one with translation group \( L_2 \).

The next lemma is included for completeness. We don’t use it here.

**Lemma 5.** For any \( i = 1, 2, 3, 4 \), the group \( G \) is generated by \( T \) and \( S_i \).

**proof.** Let \( H \) denote the subgroup generated by \( T \) and \( S_i \). The kernel of the surjective homomorphism \( G \rightarrow \overline{G} \) is \( T \). The image in \( \overline{G} \) of \( S_i \) is \( \{ \overline{r}, \overline{s} \} \), which generates \( \overline{G} \). Therefore the image of \( H \) is \( \overline{G} \). The Correspondence Theorem tells us that subgroups \( G \) that contain \( T \) correspond bijectively to subgroups of \( \overline{G} \). Both \( G \) and \( H \) contain \( T \) and have image \( \overline{G} \). Therefore they are equal. \( \square \)