Plane Crystallographic Groups with Point Group $D_1$.

This note describes discrete subgroups $G$ of isometries of the plane $P$ whose translation group $L$ is a lattice, meaning that it contains two independent vectors, and whose point group $\overline{G}$ is the dihedral group $D_1$, which consists of the identity and the reflection about the horizontal axis. We will see that there are three different types of discrete subgroups that have this point group.

As in the text, we put bars over symbols that represent elements of the point group $\overline{G}$ to avoid confusing them with the elements of $G$. So the elements of $\overline{G}$ are denoted by $\overline{1}$ and $\overline{r}$.

I. The Lattice

The lattice $L$ consists of the vectors $v$ such that $t_v$ is in $G$. As we know, if $v$ is in $L$, then $\overline{r}v$ is also in $L$.

**Proposition 1.** There are horizontal and vertical vectors $a = (a_1, 0)^t$ and $b = (0, b_2)^t$, respectively, such that $L$ is one of the two lattices

$$L_1 = \mathbb{Z}a + \mathbb{Z}b, \quad \text{and} \quad L_2 = \mathbb{Z}a + \mathbb{Z}c.$$ 

where $c = \frac{1}{2}(a + b)$.

Since $b = 2c - a$, $L_1 \subset L_2$. The lattice $L_1$ is called `rectangular' because the horizontal and vertical lines through its points divide the plane into rectangles. The lattice $L_2$ is obtained by adding to $L_1$ the midpoints of every one of these rectangles. It might be called a 'triangular' lattice.

There are two parameters in the description of $L$: the lengths of the vectors $a$ and $b$. The usual classification of discrete groups disregards these parameters, but the rectangular and isosceles lattices are considered different.

**Proof of the proposition.** Let $v = (v_1, v_2)^t$ be an element of $L$ not on either coordinate axis. Then $\overline{r}v = (v_1, -v_2)^t$ is in $L$, and so are the vectors $v + \overline{r}v = (2v_1, 0)^t$, and $v - \overline{r}v = (0, 2v_2)^t$. So $L$ contains nonzero horizontal and vertical vectors.

We choose $a_1$ to be the smallest positive real number such that $a = (a_1, 0)^t$ is in $L$. This is possible because $L$ contains a nonzero horizontal vector and it is a discrete group. The horizontal vectors in $L$ will be integer multiples of $a$. We choose $b_2$ similarly, so that the vertical vectors in $L$ are the integer multiples of $b = (0, b_2)^t$, and we let $L_1$ be the rectangular lattice $\mathbb{Z}a + \mathbb{Z}b = \{ma + nb \mid m, n \in \mathbb{Z}\}$. Then $L_1 \subset L$.

To complete the proof, we show that if $L$ contains a vector $w = (w_1, w_2)^t$ not in $L_1$, then $L = L_2$. Since the vectors $a, b$ form a basis for $\mathbb{R}^2$, $w$ will be a linear combination of the independent vectors $a$ and $b$, say $w = xa + yb = (xv_1, yv_2)^t$, with real coefficients $x$ and $y$. We write $x = m + p$ and $y = n + q$ with $m, n \in \mathbb{Z}$ and $0 \leq p, q < 1$. Then the vector $v = w - (ma + nb) = pa + qb$ is in $L$, but not in $L_1$. Say that $v = (v_1, v_2)^t$.

Then, as above, $v + \overline{r}v = (2v_1, 0)^t$ is in $L$. Since this is a horizontal vector, $2v_1$ is an integer multiple of $a_1$, and since $0 \leq v_1 < a_1$, there are only two possibilities: $v_1 = 0$ or $v_1 = \frac{1}{2}a_1$. Similarly, $v_2 = 0$ or $\frac{1}{2}b_2$. Thus $v$ is one of the four vectors $0, \frac{1}{2}a, \frac{1}{2}b$, or $\frac{1}{2}(a + b) = c$. It is not 0 because $v \not\in L_1$. It is not $\frac{1}{2}a$ because $a$ is a horizontal vector of minimal length in $L$, and similarly, it is not $\frac{1}{2}b$. Thus $v = c$, and similarly, it is not $\frac{1}{2}b$. Thus $v = c$, and $L = L_2$. 

Since $\overline{G}$ is the group of two elements $\{\overline{1}, \overline{r}\}$. The kernel of the homomorphism $\pi : G \to \overline{G}$, the set of translations $t_v$ with $v \in L$, has index 2 in $G$. Let’s denote the kernel by $H$.

$$H = \{t_v \in G \mid t_v(v) \in L\}.$$ 

There are two cosets $H$ and $Hg$ where $g$ can be any element of $G$ that isn’t in $H$. All elements of the coset $Hg$ map to $\overline{r}$ in $O_2$. (Since $H$ is a normal subgroup, the right coset $Hg$ is the same as the left coset $gH$.)
Therefore $g$ has the form $t_u r$ for some vector $u$. It is important to keep in mind that, though the product $g = t_u r$ is in $G$, we don’t know whether or not the factors $t_u$ and $r$ are in $G$.

The isometry $t_u r$ is a reflection or a glide reflection with horizontal glide line. The glide line is the line parallel to the $x$-axis that contains the point $\frac{1}{2}u$. We are free to change coordinates in the plane $P$, keeping the $x$-axis horizontal. So we may assume that the $x$-axis itself is a glide line. This changes the formulas for the elements of $G$. In the new coordinates, we will have $g = t_u r$, where $u$ is a horizontal vector. We can multiply on the left by a power of $t$, which is an element of $H$, to move $u$ into the interval $[0, a)$. Let’s assume that this has been done. So $u = ka$ with $0 \leq k < 1$.

We compute $g^2 = t_u r t_u r = t_u t_u r r$. Here $rr = 1$ and since $u$ is horizontal, $ru = u$. So $g^2 = t_2u$ is an element of $G$, and $2u$ is a horizontal vector in $L$. Then $2u = ma$ for some integer $m$, and since $u = ka$ with $0 \leq k < 1$, there are only two possibilities: $u = 0$ or $u = \frac{1}{2}a$.

### Description of the groups

Since there are two cosets, $H$ and $Hg$, the elements of $G$ are $\{t_v \mid v \in L\}$ and $\{t_v g \mid v \in L\}$.

We must distinguish the two types of lattice.

**Theorem.** Let $G$ be a discrete group of isometries of the plane whose point group is the dihedral group $D_1 = \{1, r\}$. Let $H = \{t_v \in G\}$ be the subgroup of translations in $G$. With notation as in Proposition 1, let $u = \frac{1}{2}a$ and let $g = t_u r$. Coordinates in the plane can be chosen so that,

- a) if $L = L_1$, then $G = H \cup Hr$ or $G = H \cup Hg$, and
- b) if $L = L_2$, then $G = H \cup Hr$.

**proof.** Suppose first that $L = L_2$. In this case, the vector $c = \frac{1}{2}(a + b)$ is in $L$. The isometry $g' = t_{-c}g = t_{-c+a} r = t_{-\frac{1}{2}b}r$ is in $G$. Since the vector $-\frac{1}{2}b$ is vertical, $g'$ is a reflection. When we shift the coordinates to make this a reflection about the $x$-axis, $g'$ becomes the reflection $r$. So the second possibility $g = t_u r$ is eliminated when $L = L_2$.

Next, suppose that $L = L_1$. Here the element $c$ isn’t available. The coset $Hg$ is the set of elements $t_w r$ with $w = ma + nb + c$. Then $w$ can’t be a vertical vector, so $Hg$ doesn’t contain any reflection. The two cases listed in a) are distinct when $L = L_1$. □