18.445 Pset 9

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Problem 41a:

Reducibility: The chain is irreducible iff there is one communicating class. However, for any n, we have that n and $n+1$ are in the same communicating class iff $\lambda_n, \mu_{n+1} > 0$. In particular, for the chain to be irreducible, we must have $\lambda_0 = \upsilon > 0$ and $\mu_1 = \mu > 0$. Also, for the chain to be well-defined, we must have $\lambda \geq 0$. Thus, we see that the chain is irreducible only if $\lambda \geq 0$ and $\mu, \upsilon > 0$. It is not hard to see that if these constraints are satisfied, then the chain is irreducible.

Recurrence: The question of recurrence only makes sense when the chain is irreducible, so we assume $\mu, \upsilon > 0$. The chain is recurrent iff

$$\sum_{n=1}^{\infty} \prod_{i=1}^{n} \frac{\mu_i}{\lambda_i}$$

diverges. By plugging in the values of μ_i and λ_i, this sum becomes

$$\sum_{n=1}^{\infty} \prod_{i=1}^{n} \frac{i\mu}{i\lambda + \upsilon} = \sum_{n=1}^{\infty} \prod_{i=1}^{n} \frac{\mu}{\lambda + (\upsilon/i)}.$$

Now, we split into a few cases.

First, suppose $\mu < \lambda$. Then, we have

$$\sum_{n=1}^{\infty} \prod_{i=1}^{n} \frac{\mu}{\lambda + (\upsilon/i)} < \sum_{n=1}^{\infty} \prod_{i=1}^{n} \frac{\mu}{\lambda} = \sum_{n=1}^{\infty} \left(\frac{\mu}{\lambda}\right)^n = \frac{\mu/\lambda}{1 - (\mu/\lambda)} < \infty.$$

Thus, in this case, we see that the chain is transient.

Next, suppose $\mu = \lambda$. Then, the desired sum becomes

$$\sum_{n=1}^{\infty} \prod_{i=1}^{n} \frac{i}{i + (\upsilon/\mu)}.$$

Once again, we split into two subcases.

First subcase: Suppose $\upsilon \leq \mu$. In this case, the above sum is greather or equal to

$$\sum_{n=1}^{\infty} \prod_{i=1}^{n} \frac{i}{i+1} = \sum_{n=1}^{\infty} \frac{1}{n+1}.$$

This is just the harmonic series, so it diverges. Therefore, the original series diverges, and the chain is recurrent.

Second subcase: Suppose $\upsilon > \mu$. In this subcase, we use the fact that

$$\prod_{i=1}^{n} \frac{i}{i + (\upsilon/\mu)}$$

grows asymptotically like

$$n^{-\upsilon/\mu}.$$

Therefore, the given sum will diverge iff

$$\sum_{n=1}^{\infty} n^{-\upsilon/\mu}$$

diverges. However, because $\upsilon > \mu$ in this case, we can use integration to show that it converges. It follows that when $\upsilon > \mu = \lambda$, the chan is transient.

Finally, suppose $\mu > \lambda$. In this case, there exists an N such that for $i > N$, we have $\mu > \lambda + (\upsilon/i)$. Let c equal

$$\prod_{i=1}^{N} \frac{\mu}{\lambda + (\upsilon/i)}.$$

We have that the desired sum diverges iff the following sum diverges.

$$\sum_{n=N+1}^{\infty} \prod_{i=1}^{n} \frac{\mu}{\lambda + (\upsilon/i)} = c \sum_{n=N+1}^{\infty} \prod_{i=N+1}^{n} \frac{\mu}{\lambda + (\upsilon/i)} > c \sum_{n=N+1}^{\infty} \prod_{i=N+1}^{n} 1 =$$

$$c \sum_{n=N+1}^{\infty} 1 = \infty.$$

As we can see, this sum diverges, so the chain must be recurrent in this case. To summarize, we see that the chain is recurrent iff $\mu > \lambda$ or $\lambda = \mu \geq \upsilon$.

Equilibrium Distribution: The chain admits an equilibrium distribution iff the sum

$$\sum_{n=1}^{\infty} \left(\frac{\lambda_0}{\lambda_n} \prod_{i=1}^{n} \frac{\lambda_i}{\mu_i} \right).$$

converges. Plugging in the values for λ_i and μ_i, this sum becomes

$$\sum_{n=1}^{\infty} \left(\frac{\upsilon}{n\lambda + \upsilon} \prod_{i=1}^{n} \frac{i\lambda + \upsilon}{i\mu} \right).$$

We can only have an equilibrium distribution if the chain is recurrent. As before, we split the problem into the possible cases.

First, suppose $\lambda < \mu$. Then, there exists an N such that we have $N\lambda + \upsilon < N\mu$. Let the constant c equal

$$\prod_{i=1}^{N} \frac{i\lambda + \upsilon}{i\mu}.$$

The desired sum converges iff the sum

$$\sum_{n=N+1}^{\infty} \left(\frac{\upsilon}{n\lambda + \upsilon} \prod_{i=1}^{n} \frac{i\lambda + \upsilon}{i\mu} \right) = c \sum_{n=N+1}^{\infty} \left(\frac{\upsilon}{n\lambda + \upsilon} \prod_{i=N+1}^{n} \frac{i\lambda + \upsilon}{i\mu} \right)$$

converges. However, this sum is less than

$$c \sum_{n=N+1}^{\infty} \left(1 \prod_{i=N+1}^{n} \frac{N\lambda + \upsilon}{N\mu} \right) = c \sum_{n=N+1}^{\infty} \left(\frac{N\lambda + \upsilon}{N\mu} \right)^{n-N}.$$

This sum is just a geometric series with a ratio that is smaller than 1. Therefore, it converges, and we must have an equilibrium distribution in this case.

The other possible case is $\lambda = \mu \geq \upsilon$. We rewrite the summation in question as

$$\sum_{n=1}^{\infty} \prod_{i=1}^{n} \frac{(i-1)\lambda + \upsilon}{i\lambda} = \sum_{n=1}^{\infty} \prod_{i=1}^{n} \frac{i - 1 + (\upsilon/\lambda)}{i} = \frac{\upsilon}{\lambda} \sum_{n=1}^{\infty} \prod_{i=2}^{n} \frac{i - 1 + (\upsilon/\lambda)}{i} \geq$$

$$\frac{v}{\lambda}\sum_{n=1}^{\infty}\prod_{i=2}^{n}\frac{i-1}{i}=\frac{v}{\lambda}\sum_{n=1}^{\infty}\frac{1}{n+1}.$$

The above series is the harmonic series, so it diverges. Therefore, the desired sum diverges in this case, and there is no invariant distribution.

To summarize the two cases, we see that we get an invariant distribution only when $\lambda < \mu$.

Problem 41b:

Because this process is a continuous time Markov Chain, there is an underlying discrete-time Markov chain. The question is whether we are certain that we will reach the state 0 in finite time. An easy calculation tells us that if we are in some state $n \neq 0$ at the kth step of our chain, then at the $k+1$st step, we will be at state $n+1$ with probability $\frac{\lambda}{\lambda+\mu}$ and at state $n-1$ with probability $\frac{\mu}{\lambda+\mu}$.

We note that before we reach state 0, this chain is equivalent to a random walk on a number line where the probability of moving in the positive direction at any step is $\frac{\lambda}{\lambda+\mu}$. In the first few lectures of class, we saw that this will reach 0 with certainty iff $\frac{\lambda}{\lambda+\mu} \leq \frac{1}{2}$. In particular, this means that we will reach extinction with certainty iff $\lambda \leq \mu$.

Problem 42a:

First, we note that if $\mu \leq \lambda$, then by the usual argument, we can see that there is no equilibrium distribution, as the chain will go off to infinity. Therefore, we only consider the case where $\mu > \lambda$.

The equilibrium distribution $\pi = (\pi_0, \pi_1, \ldots)$ satisfies the following system of equations.

$$\lambda\pi_0 = \mu\pi_1,$$

$$\forall i \neq 0, (\lambda+\mu)\pi_i = \lambda\pi_{i-1} + \mu\pi_{i+1},$$

$$\sum \pi_i = 1.$$

It is not hard to check that the solution of this system is $\pi_i = (1-\frac{\lambda}{\mu})\frac{\lambda^i}{\mu^i}$.

The expected length of the queue is

$$\sum_{i=0}^{\infty} i\pi_i = (1-\frac{\lambda}{\mu})\sum_{i=0}^{\infty} i\frac{\lambda^i}{\mu^i} = (1-\frac{\lambda}{\mu})\sum_{i=1}^{\infty}\sum_{j=i}^{\infty}\frac{\lambda^j}{\mu^j} =$$

$$(1-\frac{\lambda}{\mu})\sum_{i=1}^{\infty}\frac{\lambda^i/\mu^i}{1-(\lambda/\mu)} = \sum_{i=1}^{\infty}\frac{\lambda^i}{\mu^i} = \frac{\lambda}{\mu-\lambda}.$$

Problem 42b:

First, we note that if $2\mu \leq \lambda$, then by the usual argument, we can see that there is no equilibrium distribution, as the chain will go off to infinity. Therefore, we only consider the case where $2\mu > \lambda$.

The equilibrium distribution $\pi = (\pi_0, \pi_1, \ldots)$ satisfies the following system of equations.

$$\lambda\pi_0 = \mu\pi_1,$$

$$(\lambda+\mu)\pi_1 = \lambda\pi_0 + 2\mu\pi_2$$

$$\forall i \neq 0, 1, (\lambda+2\mu)\pi_i = \lambda\pi_{i-1} + 2\mu\pi_{i+1},$$

$$\sum \pi_i = 1.$$

It is not hard to check that the solution of this system is $\pi_0 = \frac{1-(\lambda/2\mu)}{1+(\lambda/2\mu)}, \pi_i = \left(\frac{1-(\lambda/2\mu)}{1+(\lambda/2\mu)}\right)\frac{\lambda^i}{2^{i-1}\mu^i}$. The expected length of the queue is

$$\sum_{i=0}^{\infty} i\pi_i = 2\left(\frac{1-(\lambda/2\mu)}{1+(\lambda/2\mu)}\right)\sum_{i=1}^{\infty} i\frac{\lambda^i}{2^i\mu^i} = 2\left(\frac{1-(\lambda/2\mu)}{1+(\lambda/2\mu)}\right)\sum_{i=1}^{\infty}\sum_{j=i}^{\infty}\frac{\lambda^j}{2^j\mu^j} =$$

$$2\left(\frac{1-(\lambda/2\mu)}{1+(\lambda/2\mu)}\right)\sum_{i=1}^{\infty}\frac{\lambda^i/(2^i\mu^i)}{1-(\lambda/2\mu)} = \frac{2}{1+(\lambda/2\mu)}\sum_{i=1}^{\infty}\frac{\lambda^i}{2^i\mu^i} = \frac{2}{1+(\lambda/2\mu)} * \frac{\lambda/2\mu}{1-(\lambda/2\mu)} =$$

$$\frac{4\lambda\mu}{4\mu^2-\lambda^2}.$$

Problem 42c:
The equilibrium distribution $\pi = (\pi_0, \pi_1, \ldots)$ satisfies the following system of equations.

$$\lambda\pi_0 = \mu\pi_1,$$

$$\forall i \neq 0, (\lambda + i\mu)\pi_i = \lambda\pi_{i-1} + (i+1)\mu\pi_{i+1},$$

$$\sum \pi_i = 1.$$

It is not hard to check that the solution of this system is $\pi_i = e^{-\lambda/\mu}\frac{\lambda^i}{\mu^i i!}$. The expected length is

$$\sum_{i=0}^{\infty} i\pi_i = e^{-\lambda/\mu}\sum_{i=0}^{\infty} i\frac{\lambda^i}{\mu^i i!} = e^{-\lambda/\mu}\sum_{i=1}^{\infty}\frac{\lambda^i}{\mu^i(i-1)!} =$$

$$\frac{\lambda e^{-\lambda/\mu}}{\mu}\sum_{i=0}^{\infty}\frac{\lambda^i}{\mu^i i!} = \frac{\lambda e^{-\lambda/\mu}}{\mu}e^{\lambda/\mu} = \frac{\lambda}{\mu}.$$

Problem 43a:
We have

$$E[S_n|\mathcal{F}_m] = E[X_1 + \ldots + X_n|X_1, \ldots, X_m] = E[X_1|X_1, \ldots, X_m] + \ldots + E[X_n|X_1, \ldots, X_m].$$

If $m \geq n$, this is just equal to $X_1 + \ldots + X_n$. On the other hand, if $m < n$, this equals

$$X_1 + \ldots + X_m + E[X_{m+1}] + \ldots + E[X_n] = X_1 + \ldots + X_m + \mu(n-m).$$

Problem 43b:
Using similar logic as in part a, we see that if $m \geq n$, then we have

$$E[S_n^2|\mathcal{F}_m] = (X_1 + \ldots + X_m)^2.$$

The more interesting case is when $m < n$. In this case, we have

$$E[S_n^2|\mathcal{F}_m] = E[(X_1 + \ldots X_n)^2|\mathcal{F}_m] = BLAHBLAHBLAH.$$

Problem 44a:
We have

$$E[W_{n+1}|\mathcal{F}_n] = E[\sum_{i=1}^{n} B_i X_i + B_{n+1}X_{n+1}|\mathcal{F}_n] = \sum_{i=1}^{n} B_i X_i + B_{n+1}E[X_{n+1}].$$

Problem 44b:
Using part a, we have $W_n = E[W_{n+1}|\mathcal{F}_n]$ iff

$$\forall n, B_{n+1}E[X_{n+1}] = 0.$$

Note that $E[X_{n+1}]$ does not depend on n. The above can only happen if we have $B_{n+1} = 0$ for all n or if $E[X_i] = 0$. However, we have $E[X_i] = 0$ iff $p = 1/2$. Together, all of this information implies the problem statement.

Problem 45a:

Suppose that after n days, the fraction of balls that is green is x. Each day, the number of balls increases by 1, so after n days, there are $n+2$ total balls. This means that we have $x(n+2)$ green balls and $(1-x)(n+2)$ red balls after n days.

In this situation, the probability of drawing a green ball is x. In this case, there will be $x(n+2)+1$ green balls and $(n+3)$ total balls. The new fraction of green balls will be $\frac{x(n+2)+1}{n+3}$.

Also, the probability of drawing a red ball is $1-x$. In this case, the fraction of red balls will become $\frac{x(n+2)}{n+3}$. Combining all of this together gives us

$$P[M_{n+1} = y | M_n = x] = \begin{cases} x & y = \frac{x(n+2)+1}{n+3} \\ 1-x & y = \frac{x(n+2)}{n+3} \\ 0 & \text{otherwise} \end{cases}.$$

Problem 45b:

Say that $M_n = x$. We have

$$E[M_{n+1}|\mathcal{F}_n] = E[M_{n+1}|M_n = x] = x\frac{x(n+2)+1}{n+3} + (1-x)\frac{x(n+2)}{n+3} =$$

$$\frac{x^2(n+2) + x + x(n+2) - x^2(n+2)}{n+3} = \frac{x(n+3)}{n+3} = x = M_n.$$