Eli H. Ross
eross@mit.edu
Alberto De Sole
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18.445 Problem Set 8

Exercise 36 Let X_t and Y_t be two independent Poisson processes with rate parameters λ and μ respectively, measuring the number of customers arriving in shops 1 and 2, respectively.

(a) What is the probability that a customer arrives in store 1 before any customer arrives in store 2?

(b) Show that $Z_t = X_t + Y_t$ is a Poisson process, and compute its rate.

(c) What is the probability that in the first four hours a total of 4 customers have arrived in the two stores?

(d) Given that exactly 4 customers have arrived at the two stores, what is the probability that the all went to store 1?

(e) Let $T_1^{(2)}$ be the time of arrival of the first customer in store 2. Then $X_{T_1^{(2)}}$ is the number of customers in store 1 at the time the first customer arrives in store 2. Find the probability distribution of the random variable $X_{T_1^{(2)}}$.

(f) Let T_1 be the first time that at least one customer has arrive in each of the two shops. Find the probability density function for T_1.

(a) The probability that the first customer arrives in store 1 at time x is $\lambda e^{-x\lambda}$. The probability that the first customer arrives in store 2 at time y is $\mu e^{-y\mu}$. Hence, the probability that the first customer arrives in store 1 at time x and the first customer arrives in store 2 at time y is $\lambda\mu e^{-x\lambda - y\mu}$. The desired probability is the following integral:

$$\int_0^\infty \int_x^\infty \lambda\mu e^{-x\lambda - y\mu}\, dy\, dx,$$

which we can evaluate as

$$\int_0^\infty \lambda e^{-x(\lambda+\mu)}\, dx = \boxed{\frac{\lambda}{\lambda+\mu}}.$$

■

(b) Let $\tau_Z > k$ be the event that a waiting time for Z_t greater than k. Note that $\mathbb{P}[\tau_Z > k] = \mathbb{P}[\tau_X > k, \tau_Y > k]$. Since X_t and Y_t are independent, this is $\mathbb{P}[\tau_X > k] \cdot \mathbb{P}[\tau_Y > k] = e^{-\lambda k} \cdot e^{-\mu k} = e^{-(\lambda+\mu)k}$. Hence, Z_t is a Poisson process with rate $\boxed{\lambda + \mu}$. ■

(c) The probability that n customers have arrived at store 1 during the 4 hours is $\frac{(4\lambda)^n e^{-4\lambda}}{n!}$ and the probability that n customers have arrived at store 2 during the 4 hours is $\frac{(4\mu)^n e^{-4\mu}}{n!}$. We sum over all of the possibilities:

$$\sum_{n=0}^{4}\left(\frac{(4\lambda)^n e^{-4\lambda}}{n!}\cdot\frac{(4\mu)^{4-n}e^{-4\mu}}{(4-n)!}\right) = 4^4\cdot e^{-4(\lambda+\mu)}\cdot\sum_{n=0}^{4}\frac{\lambda^n\mu^{4-n}}{n!(4-n)!}$$

$$= \frac{4^4}{4!}\cdot e^{-4(\lambda+\mu)}\cdot\sum_{n=0}^{4}\left(\binom{4}{n}\lambda^n\mu^{4-n}\right) = \boxed{\frac{32}{3}\cdot e^{-4(\lambda+\mu)}\cdot(\lambda+\mu)^4}.$$

■

(d) The desired conditional probability is

$$\frac{\frac{(4\lambda)^4 e^{-4\lambda}}{4!}\cdot\frac{(4\mu)^{4-4}e^{-4\mu}}{(4-4)!}}{\frac{32}{3}\cdot e^{-4(\lambda+\mu)}\cdot(\lambda+\mu)^4} = \frac{\lambda^4}{(\lambda+\mu)^4} = \boxed{\left(\frac{\lambda}{\lambda+\mu}\right)^4}.$$

■

(e) See that

$$\mathbb{P}[X_{T_1^{(2)}}=k] = \int_0^\infty\left(\mathbb{P}[X_{T_1^{(2)}}=k|T_1^{(2)}=t]\cdot\mathbb{P}[T_1^{(2)}=t]\right)dt$$

$$= \int_0^\infty\left(\mathbb{P}[X_t=k]\cdot\mathbb{P}[T_1^{(2)}=t]\right)dt = \int_0^\infty\left(\frac{(\lambda t)^k e^{-\lambda t}}{k!}\cdot\mu e^{-t\mu}\right)dt$$

$$= \frac{\lambda^k\cdot\mu}{(\lambda+\mu)^{k+1}}.$$

Thus, the probability distribution for $\mathbb{P}[X_{T_1^{(2)}}=k]$ is $\boxed{\left(\frac{\lambda}{\lambda+\mu}\right)^k\cdot\frac{\mu}{\lambda+\mu}}$ for $k=0,1,2,\ldots$, and 0 elsewhere.

■

(f) The probability that the first customer arrives in store 1 at time x is $\lambda e^{-x\lambda}$. The probability that the first customer arrives in store 2 at time y is $\mu e^{-y\mu}$. Hence, the probability that the first customer arrives in store 1 at time x and the first customer arrives in store 2 at time y is $\lambda\mu e^{-x\lambda-y\mu}$. If $T_1=t$, then either the second store to get a customer was store 1 at time t or store 2 at time t. Thus, we have

$$\mathbb{P}[T_1=t] = \int_0^t\lambda\mu e^{-t\lambda-y\mu}\,dy + \int_0^t\lambda\mu e^{-x\lambda-t\mu}\,dx$$

$$= \left(-\lambda e^{-t(\lambda+\mu)}+\lambda e^{-t\lambda}\right)+\left(-\mu e^{-t(\lambda+\mu)}+\mu e^{-t\mu}\right) = \boxed{\lambda e^{-t\lambda}+\mu e^{-t\mu}-(\lambda+\mu)e^{-(\lambda+\mu)t}}.$$

■

Exercise 37 Consider the continuous time Markov chain with state space $S = \{1, 2, 3, 4\}$ and infinitesimal generator

$$A = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 0 & -3 & 2 & 1 \\ 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

(a) Find the equilibrium distribution $\overline{\pi}$.

(b) Suppose the chain starts in state 1. What is the expected amount of time until it changes for the first time?

(c) Again assume that the chain starts in state 1. What is the expected amount of time until the chain reaches state 4?

(a) Take $\overline{\pi}A = 0$ where $\overline{\pi} = [\pi_1\ \pi_2\ \pi_3\ \pi_4]$. Expanding gives

$$-3\pi_1 + \pi_3 = \pi_1 - 3\pi_2 + 2\pi_3 = \pi_1 + 2\pi_2 - 4\pi_3 + \pi_4 = \pi_1 + \pi_2 + \pi_3 - \pi_4 = 0.$$

We have $\pi_3 = 3\pi_1$, so $\pi_1 - 3\pi_2 + 6\pi_1 = 0$, which gives $\pi_2 = \frac{7}{3}\pi_1$. Finally, $\pi_1 + \frac{7}{3}\pi_1 + 3\pi_1 - \pi_4 = 0$, so $\pi_4 = \frac{19}{3}\pi_1$. Hence, $\sum_{i=1}^{4} \pi_i = \frac{38}{3}\pi_1 = 1$, so $\pi_1 = \frac{3}{38}$. Back substitution gives the complete equilibrium distribution:

$$\boxed{\overline{\pi} = \left[\frac{3}{38}\ \frac{7}{38}\ \frac{9}{38}\ \frac{1}{2}\right]}.$$

■

(b) The change rate is 3 per unit of time. Hence, the expected amount of time until the first jump is $\boxed{\frac{1}{3}}$ of a unit of time. ■

(c) The expected time of first passage for state 4 is $\tau_1^{(4)}$. Note that $\tau^{(4)} = -\left(A^{(4)}\right)^{-1}\overline{1}$, where $A^{(4)}$ is the matrix obtained from A by erasing row and column 4. Thus, we have that

$$A^{(4)} = \begin{bmatrix} -3 & 1 & 1 \\ 0 & -3 & 2 \\ 1 & 2 & -4 \end{bmatrix}$$

so

$$-\left(A^{(4)}\right)^{-1} = \begin{bmatrix} 8/19 & 6/19 & 5/19 \\ 2/19 & 11/19 & 6/19 \\ 3/19 & 7/19 & 9/19 \end{bmatrix}.$$

Hence, $-\left(A^{(4)}\right)^{-1}\overline{1} = [1\ 1\ 1]$, so $\boxed{\tau_1^{(4)} = 1}$. ■

Exercise 38 Consider the birth and death process with $\lambda_n = 1/(n+1)$ and $\mu_n = 1$. Show that the process is recurrent and admits and equilibrium distribution. Find the equilibrium distribution.

Since there is no absorbing state, the process will be recurrent. Indeed, see that from each $n = k$, we can get to $n = k-1$ with positive parameter 1, and we can get to $n = k+1$ with positive parameter $1/(n+1)$. At 0, we will either grow or not grow. Clearly, then, we can get from any state to any other state, so the process is recurrent. There must be an equilibrium distribution because $\sum_{n=0}^{\infty} \frac{\lambda_{n-1}\ldots\lambda_0}{\mu_n\ldots\mu_1} < \infty$. To see this, note that the denominator is 1, and the numerator is $\frac{1}{n!}$; hence, the sum is $\frac{1}{e}$. Let A be the standard birth-death process infinitesimal generator, with the specific values as given in this problem. Let $\overline{\pi} = [\pi_0\ \pi_1\ \pi_2\ \ldots]$. Then, we want $\overline{\pi}A = 0$. Expanding this out gives:

$$-\lambda_0\pi_0 + \mu_1\pi_1 = 0$$

for the first term, and then

$$\lambda_{n-1}\pi_{n-1} - (\lambda_n + \mu_n)\,\pi_n + \mu_{n+1}\pi_{n+1} = 0$$

for all the other terms ($n = 1, 2, \ldots$). We substitute $\mu_n = 1$ and $\lambda_n = 1/(n+1)$ to obtain

$$-\pi_0 + \pi_1 = 0$$

and

$$\frac{1}{n}\pi_{n-1} - \left(\frac{1}{n+1} + 1\right)\pi_n + \pi_{n+1} = 0$$

for $n = 1, 2, \ldots$, so $\pi_{n+1} = \frac{n+2}{n+1}\pi_n - \frac{1}{n}\pi_{n-1}$. Inserting $\pi_0 = \pi_1$ gives $\pi_2 = \frac{1}{2}\pi_0$. Then, $\pi_3 = \frac{4}{3}\cdot\frac{1}{2}\pi_0 - \frac{1}{2}\pi_0 = \frac{1}{6}\pi_0$. Then, $\pi_4 = \frac{5}{4}\cdot\frac{1}{6}\pi_0 - \frac{1}{3}\cdot\frac{1}{2}\pi_0 = \frac{1}{24}\pi_0$. We have compelling evidence that $\pi_n = \frac{1}{n!}\pi_0$. Let's prove this using induction. We've shown the base case (up to $n = 4$, in fact); now, suppose that this holds for $n = 0, 1, 2, \ldots, m$. Then,

$$\pi_{m+1} = \frac{m+2}{m+1}\cdot\frac{1}{m!}\pi_0 - \frac{1}{m}\cdot\frac{1}{(m-1)!}\pi_0 = \pi_0\left(\frac{m+2}{m!\cdot(m+1)} - \frac{1}{(m-1)!\cdot m}\right)$$

$$= \pi_0\left(\frac{m+2}{(m+1)!} - \frac{1}{m!}\right) = \pi_0\left(\frac{(m+2)-(m+1)}{(m+1)!}\right) = \pi_0\cdot\frac{1}{(m+1)!},$$

which completes the induction. Now, we need $\sum_{n=0}^{\infty}\pi_n = \pi_0\sum_{n=0}^{\infty}\frac{1}{n!} = 1$, so $\pi_0\cdot e = 1$, so $\pi_0 = \frac{1}{e}$. Hence, the equilibrium distribution is $\overline{\pi} = [\pi_0\ \pi_1\ \pi_2\ \ldots]$ where $\boxed{\pi_n = \frac{1}{e\cdot n!}}$. ■

Exercise 39 (K&T 1.4 p.342) A new product (a “Home Helicopter” to solve the commuting problem) is being introduced. The sales are expected to be determined by both media advertising and word-of-mouth advertising. Assume that media advertising creates new customers according to a Poisson process of rate $\alpha = 1$ customer per month. For the word-of-mouth advertising, assume that each purchaser of a Home Helicopter will generate new customers at a rate $\theta = 2$ customers per month. Let X_t be the total number of Home Helicopter customers up to time t.

(a) Model X_t as a pure birth process by specifying the birth parameters λ_n, $n \neq 0$.

(b) What is the probability that exactly two helicopters are sold during the first month?

(a) There is 1 birth rate from media advertising (regardless of the number of customers), and 2 birth rate from each customer. Hence, we can model this as a pure birth process with parameter $\boxed{\lambda_n = 2n+1}$ for $n = 0, 1, 2, \ldots$ ■

(b) Using the equations given in the textbook, we have that

$$P_0'(t) = -\lambda_0 P_0(t)$$

and

$$P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t)$$

for $n = 1, 2, \ldots$ For $P_0(t)$, we have $\lambda_0 = 1$, so

$$P_0'(t) = -P_0(t)$$

which gives $P_0(t) = Ce^{-t}$. The initial condition of $P_0(0) = 1$ gives $C = 1$, so $P_0(t) = e^{-t}$. Then, since $\lambda_1 = 2 \cdot 1 + 1 = 3$, we have $P_1'(t) = -3P_1(t) + 1 \cdot e^{-t}$. Multiplying by the integrating factor e^{3t} gives

$$e^{3t} P_1(t) = \int e^{2t}\, dt = \frac{1}{2} e^{2t} + C,$$

so $P_1(t) = \frac{1}{2} e^{-t} + Ce^{-3t}$. Using the condition that $P_1(0) = 0$, we find $C = -\frac{1}{2}$, so $P_1(t) = \frac{1}{2} e^{-t} - \frac{1}{2} e^{-3t}$. Finally, we have $\lambda_2 = 2 \cdot 2 + 1 = 5$, so $P_2'(t) = -5P_2(t) + 3\left(\frac{1}{2} e^{-t} - \frac{1}{2} e^{-3t}\right)$. Multiplying by the integrating factor e^{5t} gives

$$e^{5t} P_2(t) = \int \left(\frac{3}{2} e^{4t} - \frac{3}{2} e^{2t}\right) dt = \frac{3}{8} e^{4t} - \frac{3}{4} e^{2t} + C,$$

so $P_2(t) = \frac{3}{8} e^{-t} - \frac{3}{4} e^{-3t} + Ce^{-5t}$. Using the condition that $P_2(0) = 0$, we find $C = \frac{3}{8}$, so $P_2(t) = \frac{3}{8} e^{-t} + \frac{3}{4} e^{-3t} + \frac{3}{8} e^{-5t}$. Our desired value is $P_2(1) = \boxed{\dfrac{3}{8e} - \dfrac{3}{4e^3} + \dfrac{3}{8e^5}}$. ■

Exercise 40 (K&T 1.10 p.344) Consider a pure birth process on the states $0, 1, \dots, N$, with birth parameters $\lambda_n = (N-n)\lambda$ for $n = 0, 1, \dots, N$. Suppose that $X_0 = 0$. Determine $P_n(t) = \mathbb{P}[X_t = n]$ for $n = 0, 1, 2$.

Using the equations given in the textbook, we have that

$$P_0'(t) = -\lambda_0 P_0(t)$$

and

$$P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t)$$

for $n = 1, 2, \dots$ For $P_0(t)$, we have $\lambda_0 = N\lambda$, so

$$P_0'(t) = -N\lambda P_0(t) \quad \Rightarrow \quad \frac{P_0'(t)}{P_0(t)} = -N\lambda \quad \Rightarrow \quad \ln\left(P_0(t)\right) = -N\lambda t + C$$

so $P_0(t) = Ke^{-N\lambda t}$. The initial condition of $P_0(0) = 1$ gives $K = 1$, so $\boxed{P_0(t) = e^{-N\lambda t}}$. Then, $P_1'(t) = -\lambda_1 P_1(t) + \lambda_0 P_0(t) = -(N-1)\lambda P_1(t) + N\lambda e^{-N\lambda t}$. We multiply by the integrating factor $e^{(N-1)\lambda t}$ to obtain

$$e^{(N-1)\lambda t} P_1(t) = \int N\lambda e^{-\lambda t}\, dt = -Ne^{-\lambda t} + C,$$

so $P_1(t) = -Ne^{-N\lambda t} + Ce^{-(N-1)\lambda t}$. Using the condition $P_1(0) = 0$, we find that $C = N$. Thus, $P_1(t) = N\left(e^{-(N-1)\lambda t} - e^{-N\lambda t}\right)$, or $\boxed{P_1(t) = Ne^{-N\lambda t}\left(e^{\lambda t} - 1\right)}$. Finally, we have that $P_2'(t) = -\lambda_2 P_2(t) + \lambda_1 P_1(t) = -(N-2)\lambda P_2(t) + (N-1)\lambda \cdot Ne^{-N\lambda t}\left(e^{\lambda t} - 1\right)$. We multiply by the integrating factor $e^{(N-2)\lambda t}$ to obtain

$$\begin{aligned} e^{(N-2)\lambda t} P_2(t) &= \int \left((N-1)\lambda N\left(e^{-\lambda t} - e^{-2\lambda t}\right)\right) dt \\ &= -N(N-1)e^{-\lambda t} + \frac{N(N-1)}{2} e^{-2\lambda t} + C, \end{aligned}$$

so $P_2(t) = -N(N-1)e^{-(N-1)\lambda t} + \frac{N(N-1)}{2} e^{-N\lambda t} + Ce^{-(N-2)\lambda t}$. Using the condition $P_2(0) = 0$, we find that $C = \frac{N(N-1)}{2}$. Thus, $P_2(t) = -N(N-1)e^{-(N-1)\lambda t} + \frac{N(N-1)}{2} e^{-N\lambda t} + \frac{N(N-1)}{2} e^{-(N-2)\lambda t}$. We can factor this nicely as

$$\boxed{P_2(t) = \frac{N(N-1)}{2} e^{-N\lambda t}\left(e^{2\lambda t} - 2e^{\lambda t} + 1\right)}.$$

■