18.445 Pset 7

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Problem 31a:

We have

$$P_0(t+\Delta t)-P_0(t) = P[N_t \text{ is odd }, N_{t+\Delta t} \text{ is even}]-P[N_t \text{ is even }, N_{t+\Delta t} \text{ is odd}] = P_1(t)\alpha_1\Delta t-P_0(t)\alpha_0\Delta t \Rightarrow$$

$$\frac{\partial P_0}{\partial t} = P_1(t)\alpha_1 - P_0(t)\alpha_0.$$

Similarly,

$$\frac{\partial P_1}{\partial t} = P_0(t)\alpha_0 - P_1(t)\alpha_1.$$

Problem 31b:

We have

$$P_0(0) = 1, P_1(0) = 0.$$

Using part a, we have

$$\frac{\partial P_0}{\partial t} = P_1(t)\alpha_1 - P_0(t)\alpha_0 = \alpha_1 - (\alpha_0 + \alpha_1)P_0(t) \Rightarrow$$

$$\frac{\partial P_0 e^{(\alpha_0+\alpha_1)t}}{\partial t} = \alpha_1 e^{(\alpha_0+\alpha_1)t} \Rightarrow$$

$$P_0 e^{(\alpha_0+\alpha_1)t} = \frac{\alpha_1}{\alpha_0+\alpha_1} e^{(\alpha_0+\alpha_1)t} + c \Rightarrow$$

$$P_0 = \frac{\alpha_1}{\alpha_0+\alpha_1} + ce^{-(\alpha_0+\alpha_1)t}.$$

Plugging in the initial condition gives us

$$P_0 = \frac{\alpha_1}{\alpha_0+\alpha_1} + \frac{\alpha_0}{\alpha_0+\alpha_1} e^{-(\alpha_0+\alpha_1)t}.$$

It follows that

$$P_1 = \frac{\alpha_0}{\alpha_0+\alpha_1} - \frac{\alpha_0}{\alpha_0+\alpha_1} e^{-(\alpha_0+\alpha_1)t}.$$

Problem 32a:

Suppose $ve^{tA} = v$. Then we have,

$$\frac{\partial ve^{tA}}{\partial t} = 0 \Rightarrow$$

$$ve^{tA}tA = 0 \Rightarrow$$

$$vtA = 0 \Rightarrow vA = 0.$$

Now, suppose $vA = 0$. Let P be an invertible matrix whose first column is v. We have $A = P^{-1}XP$, which implies

$$ve^{tA} = ve^{tP^{-1}XP} = vP^{-1}e^{tX}P = v.$$

Problem 32b:

The condition that $\overline{\pi}$ is invariant in time is equivalent to the statement $\overline{\pi}e^{tA} = \overline{\pi}$. However, by part a, this is equivalent to $\overline{\pi}A = 0$, as desired.

Problem 32c:

Pick some distribution π. For all $t_1 \geq 0$, we have

$$\overline{\pi}e^{t_1 A} = (\lim_{t\to\infty} \pi e^{tA})e^{t_1 A} = \lim_{t\to\infty} \pi e^{(t+t_1)A} = \overline{\pi}.$$

By part a, the desired result follows.

Problem 33a:

We have

$$P[X_\tau = i\forall\tau \leq s+t|X_\sigma = i\forall\sigma \leq s] = P[X_\tau = i\forall\tau \in [s, s+t]|X_s = i] =$$
$$P[X_{\tau-s} = i\forall\tau \in [s, s+t]|X_0 = i] = P[X_\tau = i\forall\tau \leq t|X_0 = i].$$

Problem 33b:

We have

$$P[T_1 > t+s|T_1 > s, X_0 = i] = P[X_\tau = i\forall\tau \leq s+t|X_\sigma = i\forall\sigma \leq s] = P[X_\tau = i\forall\tau \leq t|X_0 = i] = P[T_1 > t|X_0 = i].$$

Problem 33c:

We have $f(x+y) = f(x)f(y)$. Taking $g(x) = \log f(x)$, we can turn this into the equation $g(x+y) = g(x) + g(y)$. However, this is just Cauchy's functional equation, so assuming a nice condition like the continuity of g at 0, we get $g(x) = kx$ for some k. It follows that $f(x) = e^{kx}$.

Problem 33d:

Let $f(t) = P[T_1 > t|X_0 = i]$. Using part b, we have

$$f(t+s) = P[T_1 > t+s|X_0 = i] = P[T_1 > t+s|T_1 > s, X_0 = i]P[T_1 > s|X_0 = i] =$$
$$P[T_1 > t|X_0 = i]P[T_1 > s|X_0 = i] = f(t)f(s).$$

By part c, it follows that $f(t) = e^{kt}$ for some k. However, by definition, we must have $0 \leq f(t) \leq 1$ when $t \geq 0$, and this can only happen when $k < 0$. Therefore, $f(t) = e^{-\lambda t}$ for some positive λ.

Problem 33e:

We have

$$P[X_{\Delta t} = i|X_0 = i] \sim P[T_1 > \Delta t] = e^{-\lambda\Delta t} \sim 1 - \lambda\Delta t.$$

However, we also know

$$P[X_{\Delta t} = i|X_0 = i] \sim 1 - \alpha(i)\Delta t.$$

It follows that $\lambda = \alpha(i)$, as desired.

Problem 34:

It is sufficient to show that the joint probability distributions of the T_i and X_i are the correct distributions. In particular, we must show that for all n, we have

$$P[T_1 \in (t_1, t_1+\Delta t_1), ..., T_n \in (t_n, t_n+\Delta t_n), X_0 = i_0, X_1 = i_1, ..., X_n = i_n] =$$
$$\alpha(i_0)e^{-\alpha(i_0)t_1}...\alpha(i_{n-1})e^{-\alpha(i_{n-1})(t_n-t_{n-1})}\Delta t_1...\Delta t_n p_{i_0 i_1}...p_{i_{n-1}i_n}P[X_0 = i].$$

For clarity, I will only show this in the case when $n = 2$. The general case uses similar logic. We have

$$P[T_1 \in (t_1, t_1+\Delta t_1), T_2 \in (t_2, t_2+\Delta t_2), X_0 = i, X_1 = j, X_2 = k] =$$

$$P[X_2 = k | T_2 \in (t_2, t_2 + \Delta t_2), X_1 = j, T_1 \in (t_1, t_1 + \Delta t_1), X_0 = i]*$$
$$P[T_2 \in (t_2, t_2 + \Delta t_2) | X_1 = j, T_1 \in (t_1, t_1 + \Delta t_1), X_0 = i] * P[X_1 = j | T_1 \in (t_1, t_1 + \Delta t_1), X_0 = i]*$$
$$P[T_1 \in (t_1, t_1 + \Delta t_1) | X_0 = i] * P[X_0 = i] =$$
$$P[X_{t_2+\Delta t_2} = k | \forall r \text{ such that } t_1 + \Delta t_1 \leq r \leq t_2, X_r = j \text{ and } \forall s \leq t_1, X_s = i]*$$
$$P[X_{t_2+\Delta t_2} \neq j \text{ and } \forall r \text{ with } t_1 + \Delta t_1 \leq r \leq t_2, X_r = j | X_{t_1+\Delta t_1} = j, \forall s \leq t_1, X_s = i]*$$
$$P[X_{t_1+\Delta t_1} = j | \forall s \leq t_1, X_s = i] * \alpha(i) e^{-\alpha(i) t_1} \Delta t_1 * P[X_0 = i] =$$
$$P[X_{t_2+\Delta t_2} = k | X_{t_2} = j]*$$
$$P[X_{t_2+\Delta t_2} \neq j \text{ and } \forall r \text{ with } t_1 + \Delta t_1 \leq r \leq t_2, X_r = j | X_{t_1+\Delta t_1} = j]*$$
$$P[X_{t_1+\Delta t_1} = j | X_{t_1} = i] * \alpha(i) e^{-\alpha(i) t_1} \Delta t_1 * P[X_0 = i] =$$
$$P[X_{t_2 - t_1 + \Delta t_2} \neq j, X_r = j \forall r \leq t_2 - t_1 | X_{\Delta t_1} = j]*$$
$$p_{jk} p_{ik} \alpha(i) e^{-\alpha(i) t_1} \Delta t_1 P[X_0 = i] \sim$$
$$P[T_1 \in (t_2, t_2 + \Delta t_2) | X_0 = j] * p_{jk} p_{ik} \alpha(i) e^{-\alpha(i) t_1} \Delta t_1 P[X_0 = i] =$$
$$p_{jk} p_{ik} \alpha(i) e^{-\alpha(i) t_1} \alpha(j) e^{-\alpha(j) t_2} \Delta t_1 \Delta t_2 P[X_0 = i].$$

This is the desired value for the joint distribution we wanted to calculate, finishing the proof.

Problem 35:

Let $n = \min\{n | t + \Delta t < T_1 + ... + T_{n+1}\}$. Similarly, let $m = \min\{m | t < T_1 + ... + T_{m+1}\}$. We have

$$P[X_{t+\Delta t} = j | \{X_s = i_s \forall s \leq t\}] =$$
$$P[X_n = j | X_m = i_t, X_{m-1} = i_{T_1 + ... T_{m-1}}, ..., X_1 = i_{T_1}, X_0 = i_0] = P[X_n = j | X_m = i_t] =$$
$$P[X_{t+\Delta t} = j | X_t = i_t].$$

This gives us the first property of continuous-time Markov chains. Also, noting that $n - m \in \{0, 1\}$, we have

$$P[X_{t+\Delta t} = j | X_t = i_t = i] = P[X_n = j | X_m = i] = P[X_{n-m} = j | X_0 = i] =$$
$$P[X_1 = j | X_0 = i] P[n - m = 1] + P[X_0 = j | X_0 = i] P[n - m = 0].$$

Now, we split into cases depending on wether $j = i$ or not. If $j \neq i$, the second term is 0, so

$$P[X_{t+\Delta t} = j | X_t = i_t = i] = P[X_1 = j | X_1 = i] P[n - m = 1] =$$
$$p_{ij} P[T_n \in (t, t + \Delta t) | T_{n-1} = t] \sim p_{ij} \lambda \Delta t =$$
$$p_{ij} P[T_1 < \Delta t] = P[X_{\Delta t} = j | X_0 = i].$$

This proves the second and third properties of continuous Markov chains when $j \neq i$. A similar argument will work when $j = i$, and this finished the proof.