18.445 Pset 6

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Problem 26a:
We have

$$g_X(s) = \sum_{k=0}^{\infty} p_k s^k = \sum_{k=0}^{\infty} \frac{\mu^k}{k!} e^{-\mu} s^k = e^{-\mu} \sum_{k=0}^{\infty} \frac{(\mu s)^k}{k!} = e^{-\mu} e^{\mu s} = e^{-\mu(1-s)}.$$

Problem 26b:
First, we determine the probability distribution of $X+Y$. We have

$$P[X+Y=n] = \sum_{i=0}^{n} P[X=i]P[Y=n-i] = \sum_{i=0}^{n} \frac{\alpha^i}{i!} e^{-\alpha} \frac{\beta^{n-i}}{(n-i)!} e^{-\beta} = \frac{1}{n!} e^{-(\alpha+\beta)} \sum_{i=0}^{n} \binom{n}{i} \alpha^i \beta^{n-i} =$$

$$\frac{1}{n!} e^{-(\alpha+\beta)} (\alpha+\beta)^n = \frac{(\alpha+\beta)^n}{n!} e^{-(\alpha+\beta)}.$$

In particular, we see that $X+Y$ is a Poisson random variable with parameter $\alpha+\beta$. Using part a, we get

$$g_{X+Y}(s) = e^{-(\alpha+\beta)(1-s)}.$$

Problem 27:
Let X_t be the value of the Poisson process at time t. Let Y be the event that the system is surviving at time t. We have

$$P[Y_t] = \sum_{i=0}^{\infty} P[Y_t|X_t=i]P[X_t=i] = \sum_{i=0}^{\infty} \alpha^i \frac{(t\lambda)^i}{i!} e^{-t\lambda} =$$

$$e^{-t\lambda} \sum_{i=0}^{\infty} \frac{(\alpha t\lambda)^i}{i!} = e^{-t\lambda} e^{\alpha t\lambda} = e^{-(1-\alpha)t\lambda} = g_{X_t}(\alpha).$$

Problem 28:
For each of the N points, the probability that that point is in the interval $(0,1)$ is $\frac{1}{N}$. It follows that for a specific set of i points, the probability that only those points are in the interval $(0,1)$ is

$$\left(\frac{1}{N}\right)^i \left(\frac{N-1}{N}\right)^{N-i} = \frac{(N-1)^{N-i}}{N^N}.$$

Let X be the number of points in the interval $(0,1)$. We get

$$P[X=i] = \binom{N}{i} \frac{(N-1)^{N-i}}{N^N}.$$

Now, we just need to take the limit of this as $N \to \infty$. We have

$$\lim_{N\to\infty} P[X=i] = \lim_{N\to\infty} \binom{N}{i} \frac{(N-1)^{N-i}}{N^N} = \lim_{N\to\infty} \frac{N!}{i!(N-i)!} \frac{1}{(N-1)^i} \left(\frac{N-1}{N}\right)^N =$$

$$\frac{1}{i!}\lim_{N\to\infty}\frac{N!}{(N-i)!}\frac{1}{(N-1)^i}\lim_{N\to\infty}\left(1-\frac{1}{N}\right)^N =$$
$$\frac{1}{i!}\lim_{N\to\infty}\frac{N\cdot\ldots\cdot(N-i+1)}{(N-1)\cdot\ldots\cdot(N-1)}e^{-1} = \frac{1}{i!}e^{-1}.$$

Problem 29a:

By definition of W_1, we have
$$W_1 > w_1 \Leftrightarrow X_{w_1} = 0.$$
Given that $X_{w_1} = 0$, by definition of W_2, we have
$$W_2 > w_2 \Leftrightarrow X_{w_2} - X_{w_1} = 0 \text{ or } 1.$$
Together, these equations imply the result.

Problem 29b:

We have
$$P[W_1 > w_1, W_2 > w_2] = P[X_{w_1} - X_0 = 0, X_{w_2} - X_{w_1} \leq 1] =$$
$$P[X_{w_1} - X_0 = 0](P[X_{w_2} - X_{w_1} = 0] + P[X_{w_2} - X_{w_1} = 1]) =$$
$$e^{-\lambda w_1}(e^{-\lambda(w_2-w_1)} + \lambda(w_2-w_1)e^{-\lambda(w_2-w_1)}) = e^{-\lambda w_1}(1+\lambda(w_2-w_1))e^{-\lambda(w_2-w_1)} =$$
$$(1+\lambda(w_2-w_1))e^{-\lambda w_2}.$$

Problem 29c:

Differentiating the result from part b with respect to w_1, we get
$$-\lambda e^{-\lambda w_2}.$$
Differentiating this with respect to w_2 gives us
$$\lambda^2 e^{-\lambda w_2}.$$
This is the joint distribution function.

Problem 29d:

Let g be the joint distribution function of T_1, T_2. We have
$$g(t_1, t_2) = f(t_1, t_1 + t_2) = \lambda^2 e^{-\lambda(t_1+t_2)}.$$

Problem 30:

First, I'd like to point out that the problem statement gives an incorrect number of states. To see this, consider the state $(0,1)$. In this state, both machines need to undergo repairs. One has a severe failure, and the other has a regular failure. However, because machines are repaired in a first-come first-serve basis, either of the machines could be in the repair shop. Which machine needs repairing will affect the transition probabilities. Therefore, we split the state $(0,1)$ into two states. In the state $(0,1)r$, the machine with a regular failure is first in line to be repaired. In the state $(0,1)s$, the machine with a severe failure is first in line. Now, we specify the infintesimal matrix. The columns correspond to the states in the following order $(2,0),(1,0),(1,1),(0,0),(0,1)r,(0,1)s,(0,2)$. The matrix is
$$\begin{bmatrix} -2\mu & 2\mu p & 2\mu q & 0 & 0 & 0 & 0 \\ \alpha & -\alpha-\mu & 0 & \mu p & \mu q & 0 & 0 \\ \beta & 0 & -\beta-\mu & 0 & 0 & \mu p & \mu q \\ 0 & \alpha & 0 & -\alpha & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & -\alpha & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 & -\beta & 0 \\ 0 & 0 & \beta & 0 & 0 & 0 & -\beta \end{bmatrix}.$$