# 18.445 Pset 5

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**Problem 21a:**

Because $P$ is stochastic, it has the vector of all 1s as a right eigenvector with eigenvalue 1. It follows that there exists a left eigenvector $v$ with eigenvalue 1.

We have

$$\sum_j \left|(v^+P)_j\right| = \sum_j \left|\sum_i |v_i|P_{ij}\right| \le \sum_j \sum_i ||v_i|P_{ij}| = \sum_j \sum_i |v_i|P_{ij} = \sum_i \left(|v_i| \sum_j P_{ij}\right) = \sum_i |v_i| = \sum_i v_i^+.$$

Also,

$$(v^+P)_j = \sum_i |v_i|P_{ij} = \sum_i |v_iP_{ij}| \ge \left|\sum_i v_iP_{ij}\right| = |(vP)_j| = |v_j| = v_j^+$$

Together, the above two results imply that $v^+P = v^+$, which is the desired result.

**Problem 21b:**

From part a, we know that there exists a left eigenvector $v^+$ all of whose entries are positive that has eigenvalue 1. We can normalize $v^+$ to get an eigenvalue 1 eigenvector $\pi$ all of whose entries are positive and the sum of whose entries is 1. We see that $\pi$ is the desired invariant distribution.

**Problem 21c:**

Consider the 2-state Markov chain that has the identity matrix as its transition matrix. Any distribution is an invariant distribution. However, there is more than one possible initial distribution, so none of the distributions are equilibrium distributions.

**Problem 22a:**

Define $U$ to be the space of vectors $u = (u_1, \dots, u_N)$ with the property that $u_1 + \dots + u_N = 0$. This is a subspace of $\mathbb{R}^N$. For $u \in U$, we have

$$\sum_j (uP)_j = \sum_j \sum_i u_iP_{ij} = \sum_i \left(u_i \sum_j P_{ij}\right) = \sum_i u_i = 0.$$

Thus, we see that $U$ is an invariant subspace under the right-action of $P$. From problem 21, we know we have an eigenvector $v^+$ with eigenvalue 1. Because $v^+$ has non-negative entries, it is not in $U$. We can write $\mathbb{R}^N$ as $v^+ \oplus U$ and $P$ acts on each subspace separately. Therefore, to show that 1 is a simple eigenvalue of $P$, it is sufficient to show that there is no eigenvector in $U$ with eigenvalue 1.

To prove this, we first let $\epsilon > 0$ be the smallest entry in the $j_0$th column of $P$. For a vector $u \in U$, we have

$$\sum_j |(uP)_j| = \sum_j \left|\sum_i u_iP_{ij}\right| = \left(\sum_{j \ne j_0} \left|\sum_i u_iP_{ij}\right|\right) + \left|\sum_i u_iP_{ij_0}\right| = \left(\sum_{j \ne j_0} \left|\sum_i u_iP_{ij}\right|\right) + \left|\sum_i u_i(P_{ij_0} - \epsilon)\right| \le$$

$$\left(\sum_{j\neq j_0}\sum_i |u_iP_{ij}|\right)+\sum_i |u_i(P_{ij_0}-\epsilon)| = \left(\sum_{j\neq j_0}\sum_i |u_i|P_{ij}\right)+\sum_i |u_i|(P_{ij_0}-\epsilon) =$$

$$\sum_i |u_i|\left(\left(\sum_{j\neq j_0} P_{ij}\right)+P_{ij_0}-\epsilon\right) = \sum_i |u_i|(1-\epsilon) = (1-\epsilon)\sum_i |u_i|.$$

It follows that any eigenvector in $U$ must have an eigenvalue with absolute value less than 1. Therefore, there can't be any eigenvectors in $U$ with eigenvalue 1, and the desired result follows.

**Problem 22b:**
Using problems 22a and 21a, we see that the only left eigenvector of $P$ is $v^+$. The desired result follows using the same logic as in problem 21b.

**Problem 22c:**
As we saw in part $a$, for any $u \in U$, we have that

$$\sum_i |(uP)_i| \leq (1-\epsilon)\sum_i |u_i|.$$

In particular, this means that

$$\sum_i |(uP^n)_i| \leq (1-\epsilon)^n \sum_i |u_i| \Rightarrow \lim_{n\to\infty}\sum_i |(uP^n)_i| = 0 \Rightarrow \lim_{n\to\infty} uP^n = 0.$$

Every initial distribution can be expressed as $v^+ + u$ for some vector $u \in U$. We have

$$\lim_{n\to\infty}(v^+ + u)P^n = \lim_{n\to\infty} v^+P^n + \lim_{n\to\infty} uP^n = v^+ + 0 = v^+.$$

This is the desired result.

**Problem 22d:**
Let

$$P = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

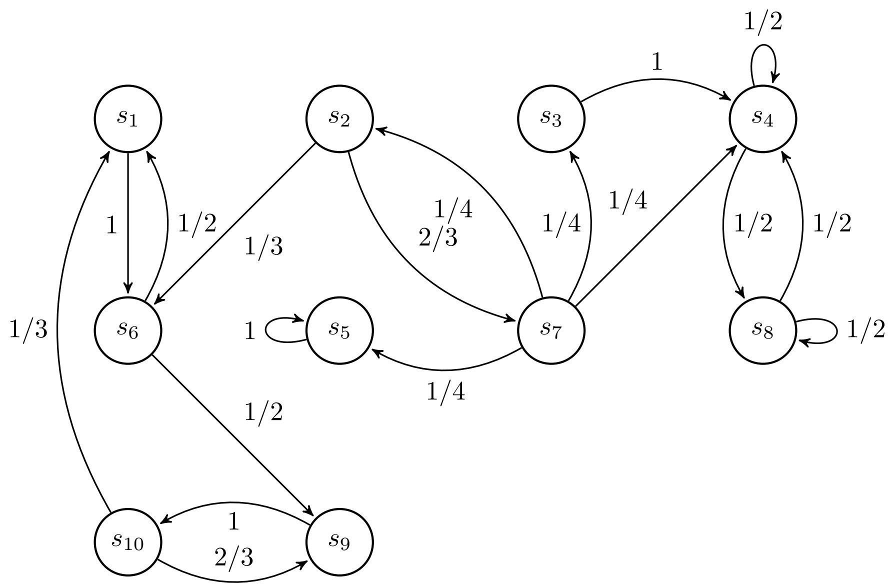
$P$ clearly satisfies the desired conditions. However, a simple calculation tells us that the equilibrium distribution is $(0, 1)$, which does not have strictly positive entries.

**Problem 22e:**
By the problem statement, for some $n$, $P^n$ is a stochastic matrix all of whose entries are strictly postitive. By the Perron Frobeious Theorem, the equilibrium distribution of $P^n$ has strictly positive entries. However, for any $\pi$, we have

$$\lim_{m\to\infty} \pi P^m = \lim_{m\to\infty} \pi (P^n)^m$$

Therefore, the equilibrium distribution of $P$ is the same as the equilibrium distribution of $P^n$, and the desired result follows.

**Problem 23a:**

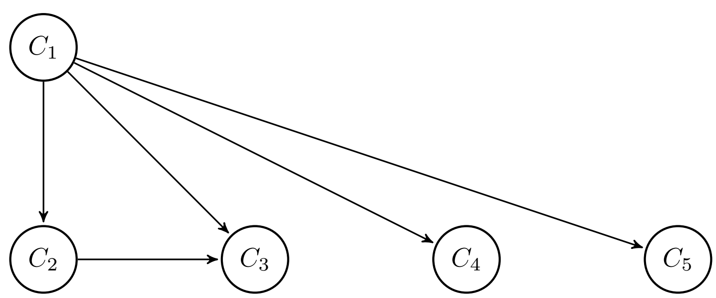



**Problem 23b:**
There are 5 communicating classes. They are

$$C_1 = \{s_2, s_7\}, C_2 = \{s_3\}, C_3 = \{s_4, s_8\}, C_4 = \{s_5\}, C_5 = \{s_1, s_6, s_9, s_{10}\}.$$

Classes $C_1, C_2$ are transient. Classes $C_3, C_4, C_5$ are recurrent. The only absorbing state is $s_5$.

**Problem 23c:**




**Problem 23d:**
The classes $C_3$ and $C_4$ are aperiodic. The class $C_5$ is periodic of period 2.

**Problem 23e:**
Rearranging the rows as $\{7, 2, 3, 1, 6, 9, 10, 4, 8, 5\}$, we get the following matrix.

$$\begin{bmatrix} 0 & 1/4 & 1/4 & 0 & 0 & 0 & 0 & 1/4 & 0 & 1/4 \\ 2/3 & 0 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & 2/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

**Problem 23f:**

Let $\pi = (\pi_4, \pi_8)$ be the equilibrium distribution in class $C_3$. Let $\alpha(i)$ be the probability of eventually getting to class $C_3$ when $X_0 = i$. We have

$$\lim_{n\to\infty} P[X_n = 4|X_0 = 2] = \pi_4 * \alpha(2).$$

By a quick symmetry argument, we know that $\pi = (1/2, 1/2)$, so $\pi_4 = 1/2$. To calculate $\alpha(2)$, we 'move' along the Markov chain. I won't go too much into the details because I think they should be relatively self-explanatory. Using the theorem of total expectation, we get the following system of equations.

$$\alpha(2) = \frac{2}{3}\alpha(7) + \frac{1}{3}\alpha(6) = \frac{2}{3}\alpha(7),$$

$$\alpha(7) = \frac{1}{4}\alpha(2) + \frac{1}{4}\alpha(5) + \frac{1}{4}\alpha(3) + \frac{1}{4}\alpha(4) = \frac{1}{4}\alpha(2) + 0 + \frac{1}{4} + \frac{1}{4} = \frac{1}{4}\alpha(2) + \frac{1}{2}.$$

Solving for $\alpha(2)$ gives us $\alpha(2) = \frac{2}{5}$. Putting all our equations together, we get that the desired answer is

$$\lim_{n\to\infty} P[X_n = 4|X_0 = 2] = \frac{1}{5}.$$

**Problem 24a:**

Let $\pi = (\pi_0, \ldots, \pi_N)$ be the invariant distribution. Let $p_{-1} = p_0 = q_N = q_{N+1} = 1$. We know that $\pi$ is the solution to the following system of equations:

$$\sum_i \pi_i = 1,$$

$$\pi_0 = q_1\pi_1,$$

$$\pi_N = p_{N-1}\pi_{N-1},$$

$$\pi_i = p_{i-1}\pi_{i-1} + q_{i+1}\pi_{i+1} \text{ for } 1 \le i \le N-1.$$

We define variables $\phi_i$ as follows.

$$\phi_i = \prod_{j=-1}^{i-1} p_j \prod_{j=i+1}^{N+1} q_j.$$

**Proposition 1.** *We have*

$$\pi_i = \frac{\phi_i}{\sum_{j=0}^N \phi_j}.$$

*Proof.* To prove the proposition, we simply show that this value of $\pi$ satisfies the given system of equations. For the first equation, we have

$$\sum_{i=0}^N \pi_i = \sum_{i=0}^N \frac{\phi_i}{\sum_{j=0}^N \phi_j} = \frac{\sum_{i=0}^N \phi_i}{\sum_{j=0}^N \phi_j} = 1.$$

By multiplying the other three equations by $\sum_{j=0}^N \phi_j$, we see that it is sufficient to show that they are satisfied when we replace $\pi_i$ by $\phi_i$.

For the second equation, we have

$$\phi_0 = \prod_{j=-1}^{-1} p_j \prod_{j=1}^{N+1} q_j = \frac{q_1}{p_0} \prod_{j=-1}^{0} p_j \prod_{j=2}^{N+1} q_j = q_1 \prod_{j=-1}^{0} p_j \prod_{j=2}^{N+1} q_j = q_1\phi_1.$$

For the third equation, we have

$$\phi_N = \prod_{j=-1}^{N-1} p_j \prod_{j=N+1}^{N+1} q_j = \frac{p_{N-1}}{q_N} \prod_{j=-1}^{N-2} p_j \prod_{j=N}^{N+1} q_j = \frac{p_{N-1}}{q_N} \prod_{j=-1}^{N-2} p_j \prod_{j=N}^{N+1} q_j = p_{N-1}\phi_{N-1}$$

For the fourth equation, we have

$$\phi_i = \prod_{j=-1}^{i-1} p_j \prod_{j=i+1}^{N+1} q_j = q_i \prod_{j=-1}^{i-1} p_j \prod_{j=i+1}^{N+1} q_j + p_i \prod_{j=-1}^{i-1} p_j \prod_{j=i+1}^{N+1} q_j =$$

$$p_{i-1} \prod_{j=-1}^{i-2} p_j \prod_{j=i}^{N+1} q_j + q_{i+1} \prod_{j=-1}^{i} p_j \prod_{j=i+2}^{N+1} q_j = p_{i-1}\phi_{i-1} + q_{i-1}\phi_{i-1}.$$

Because $\pi$ satisfies the system of equations, it must be the invariant distribution. □

The proposition gives us the desired answer.

**Problem 24b:**
Through a simple parity argument, we see that the chain is periodic of period 2.