Eli H. Ross
eross@mit.edu
Alberto De Sole
October 6, 2011

18.445 Problem Set 4

Exercise 16 Consider a Markov chain with state space $S = \{1, 2, 3\}$ and transition matrix

$$P = \begin{bmatrix} 0.2 & 0.4 & 0.4 \\ 0.1 & 0.5 & 0.4 \\ 0.6 & 0.3 & 0.1 \end{bmatrix}$$

Compute the probability that, in the long run, the chain is in state 1 (does the answer depend on the initial state X_0?). Solve this problem in two different ways:
(a) by computing the matrix P^n and letting $n \to \infty$;
(b) by finding the (unique) invariant probability distribution as a left eigenvector of P.

(a) To explicitly to calculate P^n, we first need to find the eigenvalues of P so that we may diagonalize the matrix. To find the eigenvalues λ, we set

$$\det \begin{bmatrix} 0.2-\lambda & 0.4 & 0.4 \\ 0.1 & 0.5-\lambda & 0.4 \\ 0.6 & 0.3 & 0.1-\lambda \end{bmatrix} = 0.$$

This gives

$$\begin{gathered} (0.2-\lambda)\left[(0.5-\lambda)(0.1-\lambda)-(0.3)(0.4)\right] \\ -0.4\left[(0.1)(0.1-\lambda)-(.4)(.6)\right] \\ +0.4\left[(0.1)(0.3)-(0.5-\lambda)(0.6)\right] = 0, \end{gathered}$$

and expanding/simplifying results in

$$\frac{1}{100}\left(100\lambda^3 - 80\lambda^2 - 23\lambda + 3\right) = \frac{1}{100}(\lambda-1)(10\lambda+3)(10\lambda-1).$$

Thus, the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -0.3$, and $\lambda_3 = 0.1$. We solve $Pv = \lambda v$ to find corresponding eigenvectors:

$$\lambda_1 \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \lambda_2 \Rightarrow \begin{bmatrix} 4 \\ 4 \\ -9 \end{bmatrix} \qquad \lambda_3 \Rightarrow \begin{bmatrix} 4 \\ -8 \\ 7 \end{bmatrix}$$

We then have the matrix

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.3 & 0 \\ 0 & 0 & 0.1 \end{bmatrix},$$

and we can let

$$S = \begin{bmatrix} 1 & 4 & 4 \\ 1 & 4 & -8 \\ 1 & -9 & 7 \end{bmatrix} \qquad \Rightarrow \qquad S^{-1} = \begin{bmatrix} \frac{44}{156} & \frac{64}{156} & \frac{48}{156} \\ \frac{15}{156} & -\frac{3}{156} & -\frac{12}{156} \\ \frac{13}{156} & -\frac{13}{156} & 0 \end{bmatrix}.$$

We then write $P^n = SD^nS^{-1}$. Quite clearly, we have

$$D^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (-0.3)^n & 0 \\ 0 & 0 & (0.1)^n \end{bmatrix},$$

so

$$SD^nS^{-1} = \begin{bmatrix} 1 & 4(-0.3)^n & 4(0.1)^n \\ 1 & 4(-0.3)^n & -8(0.1)^n \\ 1 & -9(-0.3)^n & 7(0.1)^n \end{bmatrix} \begin{bmatrix} \frac{44}{156} & \frac{64}{156} & \frac{48}{156} \\ \frac{15}{156} & -\frac{3}{156} & -\frac{12}{156} \\ \frac{13}{156} & -\frac{13}{156} & 0 \end{bmatrix}.$$

Expanding this gives the final value of

$$P^n = \begin{bmatrix} \frac{11}{39} + \frac{5(-0.3)^n}{13} + \frac{(0.1)^n}{3} & \frac{16}{39} - \frac{(-0.3)^n}{13} - \frac{(0.1)^n}{3} & \frac{4}{13} - \frac{4(-0.3)^n}{13} \\ \frac{11}{39} + \frac{5(-0.3)^n}{13} - \frac{2(0.1)^n}{3} & \frac{16}{39} - \frac{(-0.3)^n}{13} + \frac{2(0.1)^n}{3} & \frac{4}{13} - \frac{4(-0.3)^n}{13} \\ \frac{11}{39} - \frac{45(-0.3)^n}{52} + \frac{7(0.1)^n}{12} & \frac{16}{39} + \frac{9(-0.3)^n}{52} - \frac{7(0.1)^n}{12} & \frac{4}{13} + \frac{9(-0.3)^n}{13} \end{bmatrix}.$$

Finally, we can take the limit as $n \to \infty$ to obtain

$$P^\infty = \begin{bmatrix} \frac{11}{39} & \frac{16}{39} & \frac{4}{13} \\ \frac{11}{39} & \frac{16}{39} & \frac{4}{13} \\ \frac{11}{39} & \frac{16}{39} & \frac{4}{13} \end{bmatrix}.$$

Thus, the probability that the chain is in state 1 is $\boxed{\frac{11}{39}}$, and it does not depend on the initial state X_0. ■

(b) Let the equilibrium distribution be $\pi = (\pi_1, \pi_2, \pi_3)$. We need $\pi P = \pi$, which gives

$$.2\pi_1 + .1\pi_2 + .6\pi_3 = \pi_1$$
$$.4\pi_1 + .5\pi_2 + .3\pi_3 = \pi_2$$
$$.4\pi_1 + .4\pi_2 + .1\pi_3 = \pi_3$$

Since we must have $\pi_1 + \pi_2 + \pi_3 = 1$, we write the fourth equation as $.4(1 - \pi_3) + .1\pi_3 = \pi_3$, so $\pi_3 = \frac{4}{13}$. Now we have that $\pi_1 + \pi_2 = \frac{9}{13}$, we can write $\pi_1 = \frac{9}{13} - \pi_2$ and substitute this into the second equation, giving

$$.4\left(\frac{9}{13} - \pi_2\right) + .5\pi_2 + .3 \cdot \frac{4}{13} = \pi_2$$

which gives $\pi_2 = \frac{16}{39}$. Finally,

$$\pi_1 = \frac{9}{13} - \frac{16}{39} = \boxed{\frac{11}{39}}.$$

It does not depend on the initial state X_0. ■

Exercise 17 Consider a Markov chain with state space $S = \{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{bmatrix} 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & 3/4 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(1) Is this Markov chain irreducible and/or aperiodic?
(2) Compute (approximately) $\mathbb{P}[X_{1000} = 1 | X_0 = 2]$, $\mathbb{P}[X_{1000} = 2 | X_0 = 2]$, and $\mathbb{P}[X_{1000} = 4 | X_0 = 2]$.

(1) This Markov chain is $\boxed{\text{irreducible and not aperiodic}}$.

First, we show that the Markov chain is irreducible. Indeed, we can get from any state to any other state in some number of steps. Consider the following sequences, where each step has non-zero probability:

$$1 \to 2 \to 5 \to 1. \quad 1 \to 2. \quad 1 \to 3. \quad 1 \to 2 \to 4. \quad 1 \to 2 \to 5.$$

$$2 \to 5 \to 1. \quad 2 \to 5 \to 1 \to 2. \quad 2 \to 5 \to 1 \to 3. \quad 2 \to 4. \quad 2 \to 5.$$

$$3 \to 5 \to 1. \quad 3 \to 5 \to 1 \to 2. \quad 3 \to 5 \to 1 \to 3. \quad 3 \to 4. \quad 3 \to 5.$$

$$4 \to 1. \quad 4 \to 1 \to 2. \quad 4 \to 1 \to 3. \quad 4 \to 1 \to 2 \to 4. \quad 4 \to 1 \to 2 \to 5.$$

$$5 \to 1. \quad 5 \to 1 \to 2. \quad 5 \to 1 \to 3. \quad 5 \to 1 \to 2 \to 4. \quad 5 \to 1 \to 2 \to 5.$$

Next, we show that the Markov chain is periodic; specifically, we show that state 1 has period 3. Let $X_k = 1$; then, $X_{k+1} = 2$ or $X_{k+1} = 3$. In either case, we see that X_{k+2} must be either 4 or 5. Since $P_{41} = P_{51} = 1$, we finally see that X_{k+3} must be 1. Since $X_k = 1$ implies that $X_{k+3} = 1$, state 1 has period 3. Thus, the Markov chain is not aperiodic. ■

(2) Referring back to what we explored in the proof that state 1 has period 3, we see that if $X_0 = 2$, then X_{3n+1} will be 4 or 5, $X_{3n+2} = 1$, and X_{3n} will be 2 or 3 for all positive integers n. Thus, if $X_0 = 2$, $X_{1000} = X_{3\cdot 333+1}$ must be either 4 or 5, so

$$\boxed{\mathbb{P}[X_{1000} = 1 | X_0 = 2] = \mathbb{P}[X_{1000} = 2 | X_0 = 2] = 0}.$$

Furthermore, we know that if $X_0 = 2$, $X_{998} = X_{3\cdot 332+2} = 1$. Then,

$$\begin{aligned}\mathbb{P}[X_{1000} = 4 | X_0 = 2] &= \mathbb{P}[X_{1000} = 4 | X_{998} = 1] = P_{12} \cdot P_{24} + P_{13} \cdot P_{34} \\ &= \frac{1}{3} \cdot \frac{1}{4} + \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{12} + \frac{1}{3} = \boxed{\frac{5}{12}}.\end{aligned}$$

■

Exercise 18 (K&T 1.4 p.209) A Markov chain $X_0, X_1, X_2, \ldots$ in the state space $S = \{0, 1, 2\}$ has transition probability matrix:

$$P = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{bmatrix}.$$

Every period that the process spends in state 0 incurs a cost of \$2, every period that the process spends in state 1 incurs a cost of \$5, every period that the process spends in state 2 incurs a cost of \$3. In the long run, what is the cost per period associated with this Markov chain?

Let the equilibrium distribution be $\pi = (\pi_0, \pi_1, \pi_2)$. We need $\pi P = \pi$, which gives

$$\begin{aligned}.3\pi_0 + .5\pi_1 + .5\pi_2 &= \pi_0 \\ .2\pi_0 + .1\pi_1 + .2\pi_2 &= \pi_1 \\ .5\pi_0 + .4\pi_1 + .3\pi_2 &= \pi_2\end{aligned}$$

Since we must have $\pi_0 + \pi_1 + \pi_2 = 1$, we write the second equation as $.3\pi_0 + .5(1 - \pi_0) = \pi_0$, so $\pi_0 = \frac{5}{12}$. Now we have that $\pi_1 + \pi_2 = \frac{7}{12}$, we can write $\pi_1 = \frac{7}{12} - \pi_2$ and substitute this into the third equation, giving

$$.5 \cdot \frac{5}{12} + .4\left(\frac{7}{12} - \pi_2\right) + .3\pi_2 = \pi_2$$

which gives $\pi_2 = \frac{53}{132}$. Finally, $\pi_1 = \frac{7}{12} - \frac{53}{132} = \frac{2}{11}$. Thus, the resulting distribution is

$$\pi = \left(\frac{5}{12}, \frac{2}{11}, \frac{53}{132}\right).$$

The expected cost is then

$$2 \cdot \frac{5}{12} + 5 \cdot \frac{2}{11} + 3 \cdot \frac{53}{132} = \frac{5}{6} + \frac{10}{11} + \frac{53}{44} = \boxed{\$\frac{389}{132} \approx \$2.95}.$$

■

Exercise 19 (K&T 1.7 p.210) A Markov chain $X_0, X_1, X_2, \dots$ in the state space $S = \{0, 1, 2, 3\}$ has transition probability matrix:

$$P = \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 \\ 0 & 0.3 & 0.3 & 0.4 \\ 0 & 0 & 0.6 & 0.4 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Determine the corresponding equilibrium distribution.

Let the equilibrium distribution be $\pi = (\pi_0, \pi_1, \pi_2, \pi_3)$. We then need $\pi P = \pi$, which gives

$$.1\pi_0 + \pi_3 = \pi_0$$

$$.2\pi_0 + .3\pi_1 = \pi_1$$

$$.3\pi_0 + .3\pi_1 + .6\pi_2 = \pi_2$$

$$.4\pi_0 + .4\pi_1 + .4\pi_2 = \pi_3$$

Since we must have $\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$, we write the fourth equation as $.4(1 - \pi_3) = \pi_3$, so $\pi_3 = \frac{2}{7}$. The first equation gives $.9\pi_0 = \pi_3 = \frac{2}{7}$, so $\pi_0 = \frac{20}{63}$. The second equation gives $.2\pi_0 = .7\pi_1$, so $\pi_1 = \frac{2}{7}\pi_0 = \frac{2}{7} \cdot \frac{20}{63} = \frac{40}{441}$. Finally, we have $\pi_2 = 1 - \pi_0 - \pi_1 - \pi_3 = 1 - \frac{20}{63} - \frac{40}{441} - \frac{2}{7} = \frac{15}{49}$. Thus, the resulting distribution is

$$\boxed{\pi = \left(\frac{20}{63}, \frac{40}{441}, \frac{15}{49}, \frac{2}{7}\right)}.$$

■

Exercise 20 (K&T 1.3 p.211) A Markov chain $X_0, X_1, X_2, \dots$ in the state space $S = \{0, 1, 2, 3, 4, 5\}$ has transition probability matrix:

$$P = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

where $\alpha_i > 0$ and $\sum_i \alpha_i = 1$. Determine, in the long run, the probability of being in state 0 (does it depend on the initial state X_0?).

We will do this by finding the equilibrium distribution. Let the equilibrium distribution be $\pi = (\pi_0, \pi_1, \pi_2, \pi_3, \pi_4, \pi_5)$. We then need $\pi P = \pi$, which gives

$$\alpha_0\pi_0 + \pi_1 = \pi_0 \qquad \Rightarrow \qquad \pi_1 = \pi_0(1 - \alpha_0)$$

$$\alpha_1\pi_0 + \pi_2 = \pi_1 \quad \Rightarrow \quad \pi_2 = \pi_0(1-\alpha_0) - \alpha_1\pi_0 = \pi_0(1-\alpha_0-\alpha_1)$$
$$\alpha_2\pi_0 + \pi_3 = \pi_2 \quad \Rightarrow \quad \pi_3 = \pi_0(1-\alpha_0-\alpha_1) - \alpha_2\pi_0 = \pi_0(1-\alpha_0-\alpha_1-\alpha_2)$$
$$\alpha_3\pi_0 + \pi_4 = \pi_3 \quad \Rightarrow \quad \pi_4 = \pi_0(1-\alpha_0-\alpha_1-\alpha_2) - \alpha_3\pi_0 = \pi_0(1-\alpha_0-\alpha_1-\alpha_2-\alpha_3)$$
$$\alpha_4\pi_0 + \pi_5 = \pi_4 \;\Rightarrow\; \pi_5 = \pi_0(1-\alpha_0-\alpha_1-\alpha_2-\alpha_3) - \alpha_4\pi_0 = \pi_0(1-\alpha_0-\alpha_1-\alpha_2-\alpha_3-\alpha_4)$$
$$\alpha_5\pi_0 = \pi_5$$

We know that $\pi_0 + \pi_1 + \cdots + \pi_5 = 1$, so

$$\pi_0\left((1) + (1-\alpha_0) + (1-\alpha_0-\alpha_1) + \cdots + (1-\alpha_0-\alpha_1-\cdots-\alpha_4)\right) = 1$$

so

$$\pi_0\left(6 - 5\alpha_0 - 4\alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4\right) = 1.$$

Thus, in the long run, the probability of being in state 0 is

$$\boxed{\frac{1}{6 - 5\alpha_0 - 4\alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4}}.$$

It does not depend on the initial state X_0. We point out that, by writing $6 = 6(\alpha_0 + \alpha_1 + \cdots + \alpha_5)$, we can also express this probability as

$$\frac{1}{(6\alpha_0 + 6\alpha_1 + \cdots + 6\alpha_5) - 5\alpha_0 - 4\alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4} = \frac{1}{\alpha_0 + 2\alpha_1 + \cdots + 6\alpha_5}$$

$$= \boxed{\frac{1}{\sum_{k=0}^{5}(k+1)\alpha_k}}.$$

The solution is complete. However, as a final remark, we will also provide a possibly more intuitive method (though slightly less rigorous) to obtain this answer: note that, if there is some $X_j = 0$, then the next k steps will be $k, k-1, \ldots, 0$ with probability α_k. Consider each of these sequences of length $k+1$ to be a "block". The expected value for the length of all blocks is then $\sum_{k=0}^{5}(k+1)\alpha_k$. Since each block is in state 0 exactly once, the expected number of times that the chain is in state 0 over some long run number of steps S is equal to the expected number of chains: $\frac{S}{\sum_{k=0}^{5}(k+1)\alpha_k}$, so the probability that the chain is in state 0 in one of the given S steps is $\frac{1}{S} \cdot \frac{S}{\sum_{k=0}^{5}(k+1)\alpha_k} = \frac{1}{\sum_{k=0}^{5}(k+1)\alpha_k}$, as we obtained using the more rigorous equilibrium distribution method. ∎