18.445 Problem Set 3. Solutions

Exercise 11 (K&T 2.5 p.105) A Markov chain $X_n \in \{0, 1, 2\}$, starting from $X_0 = 0$, has the transition probability matrix

$$P = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0 & 1 \end{bmatrix}$$

Let $T = \inf\{n \geq 0 | X_n = 2\}$ be the first time that the process reaches state 2, where it is absorbed. If in some experiment we observed such a process and noted that absorption has not taken place yet, we might be interested in the conditional probability that the process is in state 0 (or 1), given that absorption has not taken place. Determine $\mathbb{P}[X_3 = 0 | T > 3]$.

We first calculate

$$P^3 = \begin{bmatrix} 0.457 & 0.23 & 0.313 \\ 0.345 & 0.227 & 0.428 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $T > 3$, we know that $X_3 = 0$ or $X_3 = 1$. We are given that $X_0 = 0$. Thus,

$$\mathbb{P}[X_3 = 0 | T > 3] = \frac{\mathbb{P}[X_3 = 0]}{\mathbb{P}[X_3 = 0] + \mathbb{P}[X_3 = 1]} = \frac{p_{00}^{(3)}}{p_{00}^{(3)} + p_{01}^{(3)}} = \frac{.457}{.457 + .23} = \boxed{\frac{457}{687}}.$$

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Exercise 12 (K&T 3.8 p.115) Two urns A and B contain a total of n balls. Assume that at time t there were exactly k balls in A. At time $t+1$ an urn is selected at random in proportion to its content (i.e. A is selected with probability $\frac{k}{n}$ and B with probability $\frac{n-k}{n}$). Then a ball is selected from A with probability p and from B with probability $q = 1 - p$ and placed in the previously chosen urn. Determine the transition probability matrix for the Markov chain X_t = number of balls in urn A at time t.

There are four possibilities if $X_t = k$:

1. If A is picked to receive and A is picked to give, $X_{t+1} = k$. This occurs with probability $\frac{k}{n} \cdot p$.
2. If A is picked to receive and B is picked to give, $X_{t+1} = k+1$. This occurs with probability $\frac{k}{n} \cdot q$.
3. If B is picked to receive and A is picked to give, $X_{t+1} = k-1$. This occurs with probability $\frac{n-k}{n} \cdot p$.
4. If B is picked to receive and B is picked to give, $X_{t+1} = k$. This occurs with probability $\frac{n-k}{n} \cdot q$.

Clearly, $k = 0$ is an absorbing state since you select A to gain a ball with probability 0; likewise, $k = n$ is an absorbing state since you always select A to gain a ball, but the ball comes from A, so there is no change. From the above probabilities, we have that $\mathbb{P}[X_{t+1} = k+1 | X_t = k] = \frac{kq}{n}$. Also, $\mathbb{P}[X_{t+1} = k+1 | X_t = k-1] = \frac{(n-k)p}{n}$. Finally, $\mathbb{P}[X_{t+1} = k | X_t = k] = \frac{kp}{n} + \frac{(n-k)q}{n} = \frac{kp+(n-k)q}{n}$. Putting this into a matrix gives:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{(n-k)p}{n} & \frac{kp+(n-k)q}{n} & \frac{kq}{n} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{(n-k)p}{n} & \frac{kp+(n-k)q}{n} & \frac{kq}{n} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{(n-k)p}{n} & \frac{kp+(n-k)q}{n} & \frac{kq}{n} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \frac{(n-k)p}{n} & \frac{kp+(n-k)q}{n} & \frac{kq}{n} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \frac{(n-k)p}{n} & \frac{kp+(n-k)q}{n} & \frac{kq}{n} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

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Exercise 13 (K&T 4.4 p.131) Consider the Markov chain $X_n \in \{0, 1, 2, 3\}$ starting with state $X_0 = 1$ and with the following transition probability matrix:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.1 & 0.2 & 0.5 & 0.2 \\ 0.1 & 0.2 & 0.6 & 0.1 \\ 0.2 & 0.2 & 0.3 & 0.3 \end{bmatrix}.$$

Determine the probability that the process never visits state 2.

Because 0 is an absorbing state, the process will eventually end up in state 0. What we want to know is whether or not the process visits state 2 before that point. To do this, we will stop the process if it visits state 2 by pretending that state 2 is an absorbing state:

$$P^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.1 & 0.2 & 0.5 & 0.2 \\ 0 & 0 & 1 & 0 \\ 0.2 & 0.2 & 0.3 & 0.3 \end{bmatrix}.$$

Then, after infinitely long time, the system will either be absorbed into state 0 or state 2. The desired probability that the process never visits state 2 is the probability that this new process is absorbed into state 0. We compute this using a first step analysis.

Let $T = \min\{n \geq 0 | X_n = 0 \text{ or } X_n = 2\}$ and $u_i = \mathbb{P}[X_T = 0 | X_0 = i]$ for $i = 1, 3$. Considering $X_0 = 1$, we obtain

$$u_1 = P_{10} + P_{11}u_1 + P_{13}u_3 = .1 + .2u_1 + .2u_3.$$

Similarly, considering $X_0 = 3$, we obtain

$$u_3 = P_{30} + P_{31}u_1 + P_{33}u_3 = .2 + .2u_1 + .3u_3.$$

Solving these equations simultaneously gives $u_1 = \frac{11}{52}$ and $u_3 = \frac{9}{26}$. Since our chain starts in state 1, the probability that it will end up in state 0 (never visiting state 2) is

$$u_1 = \boxed{\frac{11}{52}}.$$

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Exercise 14 (K&T 4.15 p.134) A simplified model for the spread of a rumor goes this way: there are $N = 5$ people in a group of friends, of which some have heard the rumor and others have not. During any single period of time two people are selected at random from the group and assumed to interact. The selection is such that an encounter between any pair of friend is just as likely as any other pair. If one of these persons has heard the rumor and the other has not, then with probability $\alpha = 0.1$ the rumor is transmitted. Let X_n be the number of friends who have heard the rumor at time n. Assuming that the process begins at time 0 with a single person knowing the rumor, what is the mean time that it takes for everyone to hear it?

If $k = 1, 2, 3, 4$ people know the rumor, and an interaction occurs, the number of people who will know the rumor will be either k or $k + 1$. The only way that a new person will learn the rumor is if, of the two people chosen to interact, one knows the rumor and one does not, and the person who knows the rumor transmits it ($\alpha = 0.1$ probability). Since there are k people who know the rumor and $5 - k$ people who do not, the number of ways to choose such a pair is $k(5 - k)$, and there are a total of $\binom{5}{2} = 10$ pairs. Thus, this probability is precisely

$$\mathbb{P}[X_{n+1} = k + 1 | X_n = k] = \frac{k(5 - k)}{10} \cdot (0.1) = \frac{k(5 - k)}{100}$$

for $k = 1, 2, 3, 4$. Then, $\mathbb{P}[X_{n+1} = k + 1 | X_n = k] = 1 - \frac{k(5-k)}{100}$. We know that $k = 5$ is an absorbing state, since if everyone knows the rumor, no more people can learn it. Thus, we have the transition probability matrix

$$P = \begin{bmatrix} \frac{24}{25} & \frac{1}{25} & 0 & 0 & 0 \\ 0 & \frac{47}{50} & \frac{3}{50} & 0 & 0 \\ 0 & 0 & \frac{47}{50} & \frac{3}{50} & 0 \\ 0 & 0 & 0 & \frac{24}{25} & \frac{1}{25} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We then calculate

$$I - Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{24}{25} & \frac{1}{25} & 0 & 0 \\ 0 & \frac{47}{50} & \frac{3}{50} & 0 \\ 0 & 0 & \frac{47}{50} & \frac{3}{50} \\ 0 & 0 & 0 & \frac{24}{25} \end{bmatrix} = \begin{bmatrix} \frac{1}{25} & -\frac{1}{25} & 0 & 0 \\ 0 & \frac{3}{50} & -\frac{3}{50} & 0 \\ 0 & 0 & \frac{3}{50} & -\frac{3}{50} \\ 0 & 0 & 0 & \frac{1}{25} \end{bmatrix}.$$

We then calculate $(I - Q)^{-1}$ as

$$(I - Q)^{-1} = \begin{bmatrix} 25 & \frac{50}{3} & \frac{50}{3} & 25 \\ 0 & \frac{50}{3} & \frac{50}{3} & 25 \\ 0 & 0 & \frac{50}{3} & 25 \\ 0 & 0 & 0 & 25 \end{bmatrix}.$$

Since we begin in state 1 (one person knows the rumor), the expected time until absorption (when everyone has heard the rumor) is the sum of the elements in row 1 of

$(I - Q)^{-1}$:

$$25 + \frac{50}{3} + \frac{50}{3} + 25 = \boxed{\frac{250}{3}}.$$

Remark: We could also solve this by considering the first step analysis equations

$$\begin{aligned} v_1 &= 1 + P_{11}v_1 + P_{12}v_2 \\ v_2 &= 1 + P_{22}v_2 + P_{23}v_3 \\ v_3 &= 1 + P_{33}v_3 + P_{34}v_4 \\ v_4 &= 1 + P_{44}v_4 + P_{45}v_5 \end{aligned}$$

where $v_i = \mathbb{E}[T|X_0 = i]$ where $T = \min\{n \geq 0 | X_n = 5\}$. Noting that $v_5 = 0$, we can solve this system to obtain $v_1 = \frac{250}{3}$. ∎

Exercise 15 (K&T 4.17 p.134) The damage $X_n \in \{0,1,2\}$ of a system subjected to wear is a Markov chain with transition probability matrix

$$P = \begin{bmatrix} 0.7 & 0.3 & 0 \\ 0 & 0.6 & 0.4 \\ 0 & 0 & 1 \end{bmatrix}.$$

The system starts in state 0 and it fails when it first reaches state 2. Let

$$T = \min\{n \geq 0 | X_n = 2\}$$

be the time of failure. Evaluate the moment generating function

$$u(s) = \mathbb{E}[s^T]$$

for $0 < s < 1$.

We again use first step analysis. Denote $u_i(s) = \mathbb{E}[s^T | X_0 = i]$. Then,

$$u_0(s) = P_{00} \cdot \mathbb{E}[s^{T+1}|X_0 = 0] + P_{01} \cdot \mathbb{E}[s^{T+1}|X_0 = 1] = P_{00} \cdot s \cdot u_0(s) + P_{01} \cdot s \cdot u_1(s)$$

$$\Rightarrow u_0(s) = s\,(.7u_0(s) + .3u_1(s))$$

and

$$u_1(s) = P_{11} \cdot \mathbb{E}[s^{T+1}|X_0 = 1] + P_{12} \cdot \mathbb{E}[s^{T+1}|X_0 = 2] = P_{11} \cdot s \cdot u_1(s) + P_{12} \cdot s \cdot 1$$

$$\Rightarrow u_1(s) = s\,(.6u_1(s) + .4)$$

This last equation gives

$$u_1(s) = \frac{.4s}{1 - .6s}.$$

Then,

$$u_0(s) = \frac{.3su_1(s)}{1 - .7s} = \frac{.3s}{1 - .7s} \cdot \frac{.4s}{1 - .6s} = \boxed{\frac{.12s^2}{(1 - .7s)(1 - .6s)}}.$$

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