18.445 Problem Set 2

Exercise 6 In class we showed that, in the nearest neighbor random walk on $\mathbb{Z}$, $\{X_n\}_{n\geq 1}$, the time T_0 of first return to 0 has the following probability distribution:

$$\mathbb{P}[T_0 = n] = \frac{2}{n-1}\binom{n-1}{n/2} p^{n/2} q^{n/2}.$$

Prove, by a direct computation, that

$$\mathbb{E}[s^{T_0}] = 1 - \sqrt{1 - 4s^2 pq}.$$

We first remark that the value given in the problem statement for $\mathbb{P}[T_0 = n]$ has a caveat: since we have to take an even number of steps to get back to 0, and it is the time of first *return*, we have specifically that $\mathbb{P}[T_0 = n] = \frac{2}{n-1}\binom{n-1}{n/2} p^{n/2} q^{n/2}$ for $n = 2k$ for $k = 1, 2, 3, \ldots$, and $\mathbb{P}[T_0 = n] = 0$ otherwise. Then, we can write

$$\mathbb{E}[s^{T_0}] = \sum_{k=1}^{\infty} \left(s^{2k} \cdot \mathbb{P}[T_0 = 2k] \right)$$

To proceed, we make the following claim:

Claim. $\frac{1}{2k-1}\binom{2k-1}{k} = 2(-4)^{k-1}\binom{1/2}{k}$.

Proof. We write

$$2\binom{1/2}{k} = 2\frac{(1/2)(-1/2)(-3/2)\cdots((2k-3)/2)}{k!} = \frac{(-1)^{k-1}}{k!}\left(\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2k-3}{2}\right)$$

$$= \frac{(-1)^{k-1}}{k!\cdot 2^{k-1}}(1\cdot 3\cdots(2k-3)) = \frac{1}{k!\cdot(-2)^{k-1}}(1\cdot 3\cdots(2k-3)).$$

Then,

$$2(-4)^{k-1}\binom{1/2}{k} = \frac{(-4)^{k-1}}{k!\cdot(-2)^{k-1}}(1\cdot 3\cdots(2k-3)) = \frac{2^{k-1}}{k!}(1\cdot 3\cdots(2k-3))$$

$$= \frac{2^{k-1}}{k!}(1\cdot 3\cdots(2k-3))\cdot\frac{2\cdot 4\cdot 6\cdots(2k-2)}{2\cdot 4\cdot 6\cdots(2k-2)}$$

$$= \frac{2^{k-1}}{k!}(1\cdot 3\cdots(2k-3))\cdot\frac{2\cdot 4\cdot 6\cdots(2k-2)}{2^{k-1}\cdot 1\cdot 2\cdot 3\cdots(k-1)} = \frac{1\cdot 2\cdot 3\cdot 4\cdots(2k-2)}{k!(k-1)!}$$

$$= \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots (2k-2)}{k!(k-1)!} \cdot \frac{2k-1}{2k-1} = \frac{1}{2k-1} \cdot \frac{(2k-1)!}{k!(k-1)!} = \frac{1}{2k-1}\binom{2k-1}{k},$$

as desired. Thus, the claim is proven.

Now, using our claim, we write

$$\mathbb{P}[T_0 = 2k] = \frac{2}{2k-1}\binom{2k-1}{k}p^k q^k = 2 \cdot 2(-4)^{k-1}\binom{1/2}{k}p^k q^k = -(-4)^k\binom{1/2}{k}p^k q^k.$$

Then,

$$\begin{aligned}\mathbb{E}[s^{T_0}] = \sum_{k=1}^{\infty}\left(s^{2k} \cdot \mathbb{P}[T_0 = 2k]\right) &= \sum_{k=1}^{\infty}\left(-s^{2k} \cdot (-4)^k\binom{1/2}{k}p^k q^k\right) \\ &= -\sum_{k=1}^{\infty}\left(\binom{1/2}{k}(-4s^2pq)^k\right).\end{aligned}$$

By the Binomial Theorem, $\sum_{k=0}^{\infty}\binom{\alpha}{k}x^k = (1+x)^\alpha$ for $|x| < 1$, so we know that

$$\sum_{k=0}^{\infty}\left(\binom{1/2}{k}(-4s^2pq)^k\right) = \sqrt{1-4s^2pq},$$

and

$$\sum_{k=0}^{\infty}\left(\binom{1/2}{k}(-4s^2pq)^k\right) = 1 + \sum_{k=1}^{\infty}\left(\binom{1/2}{k}(-4s^2pq)^k\right)$$

so

$$\sum_{k=1}^{\infty}\left(\binom{1/2}{k}(-4s^2pq)^k\right) = \sqrt{1-4s^2pq} - 1$$

. Thus,

$$\mathbb{E}[s^{T_0}] = -\sum_{k=1}^{\infty}\left(\binom{1/2}{k}(-4s^2pq)^k\right) = -\left(\sqrt{1-4s^2pq} - 1\right) = \boxed{1 - \sqrt{1-4s^2pq}}.$$

$\blacksquare$

Exercise 7 The Smiths receive the paper every morning and place it on a pile after reading it. Each afternoon, with probability 1/3, someone takes all the papers in the pile and puts them in the recycling bin. Also, if there are 5 papers in the pile, Mr. Smith (with probability 1) takes the papers to the bin. Consider the number of papers X_n in the pile in the evening of day n. Is it reasonable to model this by a Markov chain? If so, what are the state space and the transition matrix?

$\boxed{\text{Yes}}$, it is reasonable to model this by a Markov chain. Specifically we use the state space $\boxed{\{0,1,2,3,4\}}$, since there cannot be 5 papers at the end of day n because if there were 5 papers in the morning, Mr. Smith would have taken all of them to the bin during the afternoon. Let X_k be the state at the end of day n, and let $X_k \neq 4$. Then, there are two possibilities for X_{k+1}; specifically, there is a $\frac{1}{3}$ chance that $X_{k+1} = 0$, and there is a $\frac{2}{3}$ chance that $X_{k+1} = X_k + 1$. The probability of X_{k+1} being anything else is 0. In the unique case of $X_k = 4$, we note that X_{k+1} must be 0 (with probability 1) since if a fifth paper was added, Mr. Smith would automatically throw them all out. Using this, we can then construct the probability transition matrix as follows:

$$\boxed{P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \\ \frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}}.$$

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Exercise 8 Consider a Markov chain with state space $\{0,1\}$ and transition matrix

$$P = \begin{bmatrix} 1/3 & 2/3 \\ 3/4 & 1/4 \end{bmatrix}.$$

Assuming that the chain starts in state 0 at time $n = 0$, what is the probability that it is in state 1 at time $n = 2$?

We have that

$$\mathbb{P}[X_2 = 1 | X_0 = 0] = \mathbb{P}[X_1 = 0, X_2 = 1 | X_0 = 0] + \mathbb{P}[X_1 = 1, X_2 = 1 | X_0 = 0]$$

which by the Markov property is precisely

$$\mathbb{P}[X_1 = 0 | X_0 = 0] \cdot \mathbb{P}[X_2 = 1 | X_1 = 0] + \mathbb{P}[X_1 = 1 | X_0 = 0] \cdot \mathbb{P}[X_2 = 1 | X_1 = 1]$$

$$= P_{00} \cdot P_{01} + P_{01} \cdot P_{11} = \frac{1}{3} \cdot \frac{2}{3} + \frac{2}{3} \cdot \frac{1}{4} = \frac{2}{9} + \frac{2}{12} = \boxed{\frac{7}{18}}.$$

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Exercise 9 (K&T 1.5 p.99) A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix (for the states $\{0, 1, 2\}$):

$$P = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{bmatrix}$$

and initial distribution: $p_0 = 0.5$, $p_1 = 0.5$, $p_2 = 0$. Determine the probabilities:

(1) $\mathbb{P}[X_0 = 1, X_1 = 1, X_2 = 0]$,

(2) $\mathbb{P}[X_0 = 1, X_1 = 1, X_3 = 0]$.

Using the Markov property, we have that

$$\mathbb{P}[X_0 = 1, X_1 = 1, X_2 = 0] = \mathbb{P}[X_2 = 0 | X_1 = 1] \cdot \mathbb{P}[X_1 = 1 | X_0 = 1] \cdot \mathbb{P}[X_0 = 1]$$

$$= P_{10} \cdot P_{11} \cdot p_0 = 0.5 \cdot 0.1 \cdot 0.5 = \boxed{0.025}.$$

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Using the Markov property, we have that

$$\mathbb{P}[X_0 = 1, X_1 = 1, X_3 = 0]$$

$$\begin{aligned} &= \mathbb{P}[X_0 = 1, X_1 = 1, X_2 = 0, X_3 = 0] \\ &+ \mathbb{P}[X_0 = 1, X_1 = 1, X_2 = 1, X_3 = 0] \\ &+ \mathbb{P}[X_0 = 1, X_1 = 1, X_2 = 2, X_3 = 0] \end{aligned}$$

$$\begin{aligned} &= \mathbb{P}[X_3 = 0 | X_2 = 0] \cdot \mathbb{P}[X_2 = 0 | X_1 = 1] \cdot \mathbb{P}[X_1 = 1 | X_0 = 1] \cdot \mathbb{P}[X_0 = 1] \\ &+ \mathbb{P}[X_3 = 0 | X_2 = 1] \cdot \mathbb{P}[X_2 = 1 | X_1 = 1] \cdot \mathbb{P}[X_1 = 1 | X_0 = 1] \cdot \mathbb{P}[X_0 = 1] \\ &+ \mathbb{P}[X_3 = 0 | X_2 = 2] \cdot \mathbb{P}[X_2 = 2 | X_1 = 1] \cdot \mathbb{P}[X_1 = 1 | X_0 = 1] \cdot \mathbb{P}[X_0 = 1] \end{aligned}$$

$$= P_{00} \cdot P_{10} \cdot P_{11} \cdot p_0 + P_{10} \cdot P_{11} \cdot P_{11} \cdot p_0 + P_{20} \cdot P_{12} \cdot P_{11} \cdot p_0$$

$$= p_0 \cdot P_{11} \cdot (P_{00} \cdot P_{10} + P_{10} \cdot P_{11} + P_{20} \cdot P_{12}) = 0.5 \cdot 0.1 \cdot (0.3 \cdot 0.5 + 0.5 \cdot 0.1 + 0.5 \cdot 0.4)$$

$$= 0.5 \cdot 0.1 \cdot (.15 + .05 + .2) = 0.5 \cdot 0.1 \cdot 0.4 = \boxed{0.02}.$$

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Exercise 10 (K&T 1.4 p.100) The random variables $\xi_1, \xi_2, \ldots$ are independent identically distributed, with common probability distribution

$$\mathbb{P}[\xi = 0] = 0.1, \;\; \mathbb{P}[\xi = 1] = 0.3, \;\; \mathbb{P}[\xi = 2] = 0.2, \;\; \mathbb{P}[\xi = 3] = 0.4.$$

Set $X_0 = 0$ and $X_n = \max(\xi_1, \ldots, \xi_n)$ be the largest ξ observed to date. Determine the transition probability matrix for the Markov chain $\{X_n\}$.

We first note that $X_{k+1} \geq X_k$ for all k, since if $\xi_{k+1} \leq X_k$, $X_{k+1} = X_k$, and if $\xi_{k+1} > X_k$, then $X_{k+1} = \xi_{k+1} > X_k$. Thus, $P_{ab} = 0$ if $a > b$, so

$$P_{10} = P_{20} = P_{21} = P_{30} = P_{31} = P_{32} = 0.$$

In the cases where $a > b$, we have simply that

$$\mathbb{P}[X_{k+1} = a | X_k = b] = \mathbb{P}[\xi = a],$$

so

$$P_{01} = 0.3 \qquad P_{02} = P_{12} = 0.2 \qquad P_{03} = P_{13} = P_{23} = 0.4.$$

Finally, we note that

$$\mathbb{P}[X_{k+1} = a | X_k = a] = \sum_{j=0}^{a} \mathbb{P}[\xi = j]$$

so

$$\begin{gathered} P_{00} = P[\xi = 0] = 0.1 \\ P_{11} = \mathbb{P}[\xi = 0] + \mathbb{P}[\xi = 1] = 0.1 + 0.3 = 0.4 \\ P_{22} = \mathbb{P}[\xi = 0] + \mathbb{P}[\xi = 1] + \mathbb{P}[\xi = 2] = 0.1 + 0.3 + 0.2 = 0.6 \\ P_{33} = \mathbb{P}[\xi = 0] + \mathbb{P}[\xi = 1] + \mathbb{P}[\xi = 2] + \mathbb{P}[\xi = 3] = 0.1 + 0.3 + 0.2 + 0.4 = 1 \end{gathered}$$

Thus, we can construct the transition probability matrix

$$\boxed{P = \begin{bmatrix} 0.1 & 0.3 & 0.2 & 0.4 \\ 0.0 & 0.4 & 0.2 & 0.4 \\ 0.0 & 0.0 & 0.6 & 0.4 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}}.$$

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