18.445. Problem Set 1. Solutions.

1. **KT 1.1 on p.61.** I roll a six-sided die and observe the number N on the upper face. I then roll a fair coin N times and I observe heads appear X times.

   Solution. i) $P\{N=3, X=2\} = P\{N=3\}P\{X=2 \mid N=3\} = \frac{1}{6} \cdot \binom{3}{2} \left(\frac{1}{2}\right)^3 = \frac{1}{16}$.

   ii) $P\{X=5\} = P\{X=5, N=5\} + P\{X=5, N=6\} = \frac{1}{6}\left(\frac{1}{2}\right)^5 + \frac{1}{6}\binom{6}{5}\left(\frac{1}{2}\right)^6 = \frac{1}{48}$.

   iii) $E[X] = \sum_{i=1}^{6} E[X \mid N=i]P\{N=i\} = \frac{1}{6}\sum_{i=1}^{6} E[X \mid N=i] = \frac{1}{6}\sum_{i=1}^{6}\frac{i}{2} = \frac{7}{4}$.

2. **KT 1.4 on p.62.** A six-sided die is rolled and the number N on the upper face is recorded. From a jar containing 10 tags numbered $1, 2, \ldots, 10$ we select N tags at random without replacement. Let X be the smallest number on the tags drawn. Find $P[X=2]$.

   Solution. Observe that $P\{X=2 \mid N=i\} = \frac{\binom{8}{i-1}}{\binom{10}{i}} = \frac{i(10-i)}{90}$, because there are $\binom{10}{i}$ ways to choose the tags, and of these $\binom{9}{i-1}$ contain two, and of those $\binom{8}{i-1}$ do not contain a one. Hence

$$P\{X=2\} = \sum_{i=1}^{6} P\{X=2 \mid N=i\}P\{N=i\} = \frac{1}{6}\sum_{i=1}^{6}\frac{i(10-i)}{90} = \frac{119}{540}.$$

3. **KT 1.2 on p.62.** A card is picked at random from N cards labeled $1, 2, \ldots, N$, and the number that appears is X. A second card is picked at random from the cards numbered $1, 2, \ldots, X$ and its number is Y. Determine the conditional distribution of X given that $Y = y$.

   Solution. Note that

$$P\{X=x, Y=y\} = P\{Y=y \mid X=x\}P\{X=x\} = \frac{1}{Nx}$$

   and by the Bayes Rule

$$P\{Y=y\} = \sum_{i=1}^{N} Pr\{Y=y, X=i\}Pr\{X=i\} = \frac{1}{N}\sum_{i=y}^{N} Pr\{Y=y, X=i\} = \frac{1}{N}\sum_{i=y}^{N}\frac{1}{i}.$$

   Thus if $x < y$ then $P\{X=x \mid Y=y\} = 0$ and if $x \geq y$ then

$$P\{X=x \mid Y=y\} = \frac{P\{X=x, Y=y\}}{P\{Y=y\}} = \frac{1}{x} \cdot \frac{1}{\frac{1}{y} + \cdots + \frac{1}{N}}.$$

4. **KT 1.5 on p.63** A nickel is tossed 20 times in succession. Every time that the nickel comes up heads, a dime is tossed. Let X be the number of heads appearing on the tosses of the dime. Determine $P[X=0]$.

Solution. Observe that we have 20 independent events: tossing of a nickel with a possible toss of a dime after that. In order to ensure that $X = 0$, each nickel flip must either result in a tails ($p = 0.5$) or a heads followed by a dime tails ($p = 0.25$), giving a net of $p = \frac{3}{4}$ per flip. Thus, overall we have $P\{X = 0\} = \left(\frac{3}{4}\right)^{20}$.

5. **KT 1.3 on p.100.** Consider a sequence of items from a production process, with each item being graded as good or defective. Suppose that a good item is followed by another good item with probability α and is followed by a defective item with probability $1 - \alpha$. Similarly a defective item is followed by another defective item with probability β and is followed by a good item with probability $1 - \beta$. If the first item is good, what is the probability that the first defective item to appear is the fifth one?

   Solution. The first defective item appears at the fifth one, is equivalent to the first to the fourth items are good, while fifth one is defective. Therefore

$$\begin{aligned} P\{\text{fifth one is first defective}\} &= P\{1^{st} \text{ is good}\}P\{2^{nd} \text{ is good} \mid 1^{st} \text{ is good}\}\cdots \\ &\qquad \cdots P\{5^{th} \text{ is defective} \mid 4^{th} \text{ is good}\} \\ &= \alpha^3(1-\alpha). \end{aligned}$$