18.445 Pset 10

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Problem 46a:
We have

$$\lim_{N\to\infty} E[|X|\mathbf{1}_{X>N}] = \lim_{N\to\infty} \sum_{x=N+1}^{\infty} xP[|X|=x] =$$

$$E[|X|] - \lim_{N\to\infty} \sum_{x=0}^{N} xP[|X|=x] = 0.$$

Note that we can only do the last few equalities because $E[|X|] < \infty$.

Problem 46b:
We denote the compliment of A by A^c. We have

$$P[A] \le P[|X| > N] \Rightarrow$$

$$P[A, |X| > N] + P[A, |X| \le N] \le P[A, |X| > N] + P[A^c, |X| > N] \Rightarrow$$
$$P[A, |X| \le N] \le P[A^c, |X| > N].$$

Now, using this we have

$$E[|X|\mathbf{1}_{A,|X|\le N}] \le NE[\mathbf{1}_{A,|X|\le N}] = NP[A, |X| \le N] \le$$

$$NP[A^c, |X| > N] = NE[\mathbf{1}_{A^c,|X|>N}] \le E[|X|\mathbf{1}_{A^c,|X|>N}].$$

Finally, starting with the inequality we just derived, we have

$$E[|X|\mathbf{1}_{A,|X|\le N}] \le E[|X|\mathbf{1}_{A^c,|X|>N}] \Rightarrow$$

$$E[|X|\mathbf{1}_{A,|X|\le N}] + E[|X|\mathbf{1}_{A,|X|>N}] \le E[|X|\mathbf{1}_{A^c,|X|>N}] + E[|X|\mathbf{1}_{A,|X|>N}] \Rightarrow$$
$$E[|X|\mathbf{1}_{A}] \le E[|X|\mathbf{1}_{|X|>N}].$$

This is the desired result.

Problem 46c:
Pick some $\epsilon > 0$. We would like to show that there exists an M such that for all $m > M$, we have $E[|X|\mathbf{1}_{A_M}] < \epsilon$. However, because $E[|X|] < \infty$, there must exist an N_1 such that $E[|X|\mathbf{1}_{|X|>N_1}] < \epsilon$. Next, we use the fact that

$$\lim_{N\to\infty} P[A_N] = 0$$

to see that there must exist an M such that for $m > M$, we have

$$P[A_m] \le P[X > N_1].$$

By part b, it follows that for $m > M$, we have

$$E[|X|\mathbf{1}_{A_M}] \le E[|X|\mathbf{1}_{|X|>N_1}] < \epsilon.$$

This gives us the desired result.

Problem 46d:
We are given that $E[|M_T|] = \infty$. Also, because $P[T < \infty] = 1$, we have that

$$\lim_{n\to\infty} P[T > N] = 0.$$

Setting $X = M_T$ and $A_N = \{T > N\}$, we can directly use the result from part c to get that

$$\lim_{N\to\infty} E[|M_T|\mathbf{1}_{T>N}] = 0,$$

as desired.

Problem 47a:
We have

$$E[M_{n+1}|M_n = x^2 - n] = E[M_{n+1}|X_n = x] = \frac{1}{2}((x+1)^2 - (n+1)) + \frac{1}{2}((x-1)^2 - (n+1)) =$$

$$\frac{1}{2}(x^2 + 2x + 1 - n - 1 + x^2 - 2x + 1 - n - 1) = \frac{1}{2}(2x^2 - 2n) = x^2 - n = M_n.$$

This is the desired result.

Problem 47b:
Because finite random walks are recurrent, it is not hard to see that $P[T < \infty] = 1$.
Before we continute, we prove a small lemma.

Lemma 1.

$$E[T] < \infty.$$

Proof. Let $p = 1/2$. At any point during the random walk, there is always a small p^N chance that the next N steps in the walk will be right. If this happens, we will clearly hit one of the barriers. Therefore, we have that

$$P[T \geq n + N|T \geq n] \leq 1 - p^N.$$

In particular, this means that

$$P[T = t] \leq \left(1 - p^N\right)^{\lfloor t/N \rfloor}.$$

This allows us to bound $E[T]$. In particular, we have

$$E[T] = \sum_{t=0}^{\infty} tP[T = t] \leq \sum_{t=0}^{\infty} t\left(1 - p^N\right)^{\lfloor t/N \rfloor} \leq$$

$$\sum_{t=0}^{\infty} N\left(\left\lfloor \frac{t}{N} \right\rfloor + 1\right)\left(1 - p^N\right)^{\lfloor t/N \rfloor} = N\sum_{t=0}^{\infty} N(t+1)\left(1 - p^N\right)^t =$$

$$N^2 \sum_{t=0}^{\infty}\sum_{j=t}^{\infty}(1 - p^N)^t = N^2 \sum_{t=0}^{\infty} \frac{(1 - p^N)^t}{1 - (1 - p^N)}.$$

The last sum is a geometric series, so it converges to a finite value. Therefore, we must have that $E[T]$ is finite, as desired. □

Now, we note that $|M_K| \leq N^2 + K$. Therefore, using the lemma, we have

$$E[|M_T|] \leq E[N^2 + T] \leq E[N^2] + E[T] < N^2 + \infty = \infty.$$

Therefore, by problem 1, we must have

$$\lim_{K\to\infty} E[|M_T|\mathbf{1}_{T>K}] = 0.$$

Finally, we notice that when $T > K$, we have $|M_K| < |M_T| + N^2$. Therefore, we must have

$$\lim_{K\to\infty} E[|M_K|\mathbf{1}_{T>K}] < \lim_{K\to\infty} E[(|M_T| + N^2)\mathbf{1}_{T>K}] =$$

$$\lim_{K\to\infty} E[(|M_T|)\mathbf{1}_{T>K}] + \lim_{K\to\infty} N^2 E[\mathbf{1}_{T>K}] = 0 + 0 = 0.$$

Thus, we see that all the assumptions of the Optional Sampling Theorem are fulfilled.

Problem 47c:

By the Optional Sampling Theorem, we have

$$E[M_T] = M_0 = a^2.$$

At the same time, we have

$$E[M_T] = E[X_T^2 - T] = E[X_T^2] - E[T] = N^2 * P[X_T = N] - E[T] =$$

$$aN - E[T].$$

Combining these equations gives us

$$E[T] = a(N - a).$$

Problem 48:

We prove the claim using induction on n. The base case is easy to check, so we just do the induction hypothesis. Suppose the claim is true for $n = k$. Then, we have

$$P[M_{k+1} = \frac{a+1}{k+3}] =$$

$$P[M_{k+1} = \frac{a+1}{k+3}|M_k = \frac{a+1}{k+2}] * P[M_k = \frac{a+1}{k+2}] + P[M_{k+1} = \frac{a+1}{k+3}|M_k = \frac{a}{k+2}] * P[M_k = \frac{a}{k+2}] =$$

$$\frac{k+2-(a+1)}{k+2} * \frac{1}{k+1} + \frac{a}{k+2} * \frac{1}{k+1} =$$

$$\frac{k+1}{k+2} * \frac{1}{k+1} = \frac{1}{k+2}.$$

This is the desired result.

Problem 49a:

We want to show that $E[M_{n+1}|M_n] = M_n$. If X_n has been absorbed into 0 or N, we have that $X_{n+1} = X_n \Rightarrow M_{n+1} = M_n$. Therefore, after X_n is absorbed, M_n acts as a martingale. Now, we must just show that M_n acts as a martingale before X_n is absorbed. We have

$$E[M_{n+1}|M_n = \left(\frac{1-p}{p}\right)^x] = E[\left(\frac{1-p}{p}\right)^{X_n+1}|X_n = x] =$$

$$p\left(\frac{1-p}{p}\right)^{x+1} + (1-p)\left(\frac{1-p}{p}\right)^{x-1} = \left(\frac{1-p}{p}\right)^x = M_n.$$

This gives us the desired result.

Problem 49b:

Modelling X_n as a Markov chain, we see that the only recurrent states are the absorbing states. It follows that $P[T < \infty] = 1$. Also, because X_n is bounded, we must also have that M_n is bounded. Combining this fact with $P[T > \infty] = 1$ gives us the last two conditions of the Optional Sampling Theorem.

Problem 49c:

By the Optional Sampling Theorem, we have that

$$E[|M_T|] = M_0 = \left(\frac{1-p}{p}\right)^a.$$

On the other hand, we also have

$$E[|M_T|] = p_0\left(\frac{1-p}{p}\right)^0 + (1-p_0)\left(\frac{1-p}{p}\right)^N.$$

Setting these equal to each other gives

$$p_0 = \frac{\left(\frac{1-p}{p}\right)^a - \left(\frac{1-p}{p}\right)^N}{1-\left(\frac{1-p}{p}\right)^N}.$$

Problem 49d:

We have

$$E[W_{n+1}|W_n = x+(1-2p)n] = E[W_{n+1}|X_n = x] = p*((x+1)+(1-2p)(n+1))+(1-p)*((x-1)+(1-2p)(n+1)) =$$

$$x-(1-2p)+(1-2p)(n+1) = x+(1-2p)n = W_n.$$

Thus, W_n is a martingale, as desired.

Problem 49e:

At every point on the walk, there is a greater than p^N chance that the walk will go right until it gets to a point greater than or equal to N. It follows that $P[T < \infty] = 1$.

By using logic that is similar to the logic of Lemma 1 of problem 47b, we also see that $E[T] < \infty$. From here, we can just follow the same logic as we used in problem 47b to get the desired result.

Problem 49f:

By the Optional Sampling Theorem, we have

$$E[W_T] = W_0 = a.$$

At the same time, we have

$$E[W_T] = E[X_T + (1-2p)T] = E[X_T] + E[(1-2p)T] = (1-p_0)N + (1-2p)E[T].$$

Here, p_0 is the value we calculated in part c. Putting these together gives us

$$E[T] = \frac{(1-p_0)N - a}{1-2p}.$$

Problem 50a:

First, we note that if $n \geq N$, then we have that $N_n = N_{n+1} = N$, which implies that $M_{n+1} = M_n$. Clearly, in this case we have $E[M_{n+1}|M_n] = M_n$.

Now, say that $n < N$ and $n \neq 0$. In this case, $N_n = n$. We have

$$E[M_{n+1}|M_n = a(x)] = E[M_{n+1}|X_n = x] =$$

$$a(x+1)P[X_{n+1}=x+1|X_n=x]+a(x-1)P[X_{n+1}=x-1|X_n=x]=$$

$$a(x+1)\frac{\lambda_n}{\lambda_n+\mu_n}+a(x-1)\frac{\mu_n}{\lambda_n+\mu_n}=a(x)=M_n.$$

If $n=0$, then similar logic as above will work in showing that $E[M_1|M_0]=M_0$. Together, all of the above statements prove that M_n is a martingale.

Problem 50b:

We must show that $E[|M_n|]$ is bounded. However, this is not hard to see as $0 \leq M_n \leq 1$ for all n.

Problem 50c:

The 'if' direction of the theorem is not difficult, so we concentrate on the other direction.

Suppose that X_n is recurrent. Because X_n is recurrent, we have that N is finite. In particular, this means that after a certain point, we will always have $n > N$, which implies that $M_n = a(X_N) = 1$. In particular, we see that

$$E[M_0] = E[M_\infty] = 1.$$

It follows that $E[a(X_0)] = 1$. However, because $a(n) \leq 1$, this means that $a(X_0)$ is 1 no matter what X_0 is. It follows that $a(n) = 1$ for all n, as desired.