Stochastic Processes – 18.445

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Practice Mid Term Exam 2

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Problem 1: Let N_t be a birth/death process with: $\lambda_n = 4$ for $n \neq 8$, $\lambda_8 = 0$, $\mu_n = 5$ for $n \neq 9$ and $\mu_9 = 0$.

(a) Describe the communicating classes.

(b) Assuming $N_0 \leq 8$, is the process recurrent or transient? In the forrmer case find, if it exists, the equilibrium distribution. In the latter case, find if explosion occurs with positive probability.

(c) Assuming $N_0 \geq 9$, is the process recurrent or transient? In the forrmer case find, if it exists, the equilibrium distribution. In the latter case, find if explosion occurs with positive probability.

Solution: (a) There are two communicating classes: $\{0, 1, \dots, 8\}$ and $\{9, 10, 11, \dots\}$.

(b) Since the class $\{0, 1, \dots, 8\}$ is finite, if $N_0 \leq 8$ the process becomes an irreducible continuous time Markov chain on a finite state space, with infinitesimal generator matrix

$$A = \begin{pmatrix} -4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & -9 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & -9 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & -9 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & -9 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & -9 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & -9 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & -9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -5 \end{pmatrix} .$$

Hence, by the Perron-Frobenious Theorem, it is recurrent and it admits a (unique) equilibrium limiting distribution $\bar{\pi} = (x_0, x_1, \dots, x_8)$, solution of $\bar{\pi}A = 0$. Solving the system of equations we get:

$$x_8 = \frac{4}{5}x_7 \, , \; x_7 = \frac{4}{5}x_6 \, , \; \dots \, , \; x_1 = \frac{4}{5}x_0 \, ,$$

namely

$$x_n = \Big(\frac{4}{5}\Big)^n x_0 \ , \quad n = 0, \dots, 8 \, ,$$

and imposing the normalization condition $x_0 + \dots + x_9 = 1$, we get

$$x_0 = \frac{1}{\sum_{n=0}^{8} \left(\frac{4}{5}\right)^n} = \frac{5^8}{5^9 - 4^9} \, .$$

Hence,

$$x_n = \frac{4^n 5^{8-n}}{5^9 - 4^9} \ , \quad n = 0, \dots, 8 \, .$$

(c) For $N_0 \geq 9$, the process is the same as a MM1 queueing model with $\lambda_n = 4$, $\mu_n = 5$ for every n, shifted (by 9). Since the series $\sum_n \frac{\mu_{n+9} \dots \mu_10}{\lambda_{n+9} \dots \lambda 10} = \sum_n \left(\frac{5}{4}\right)^n$ is divergent, we know that the process is recurrent. In fact, since the series $\sum_n \frac{\lambda_{n-1+9} \dots \lambda_9}{\mu_{n+9} \dots \mu 10} = \sum_n \left(\frac{4}{5}\right)^n$ is convergent, the process is positive recurrent, and it admits a unique limiting equilibrium distribution $\bar{\pi}$. It is given by

$$\bar{\pi}_{n+9} = \frac{\frac{\lambda_{n-1+9} \dots \lambda_9}{\mu_{n+9} \dots \mu 10}}{\sum_{m=0}^{\infty} \frac{\lambda_{m-1+9} \dots \lambda_9}{\mu_{m+9} \dots \mu 10}} = \frac{\left(\frac{4}{5}\right)^n}{\sum_{m=0}^{\infty} \left(\frac{4}{5}\right)^m} = \frac{4^n}{5^{n+1}} \, .$$

Problem 2: A small barber shop, operated by a single barber, has room for only two costumers. Potential costumers arrive at a Poisson rate of 3 per hour, and the successive serving times are independent exponential random variables of mean 1/4 hour.

(a) What is the average number of costumers in the shop?

(b) What is the proportion of potential costumers that enter the shop?

(c) If the barber could work twice as fast, how much more business would he do (in average)?

Solution: (a) We model this situation with a continuous time Markov chain X_t, describing the number of customers in the shop at time t, which takes the possible values $0, 1, 2$. If $X_t < 2$ the process can make a jump of $+1$ with rate 3/hour, while if $X_t > 0$ the process can make a jump of -1 with rate 4/hour (Recall: the mean value of an exponential random variable of rate λ is $1/\lambda$). Hence, the infinitesimal generator matrix of this process is:

$$A = \left(\begin{array}{lll} -3 & 3 & 0 \\ 4 & -7 & 3 \\ 0 & 4 & -4 \end{array} \right) .$$

The corresponding equilibrium distribution $\bar{\pi}$ is solution of $\bar{\pi} A = 0$. Solving the system of equations we get:

$$\bar{\pi}_1 = \frac{3}{4}\bar{\pi}_0 \,,\ \bar{\pi}_2 = \frac{3}{4}\bar{\pi}_1 \,,$$

and normalizing, we get

$$\bar{\pi}_0 = \frac{4^2}{4^3 - 3^3} \ ,\ \ \bar{\pi}_1 = \frac{3 \cdot 4}{4^3 - 3^3} \ ,\ \ \bar{\pi}_2 = \frac{3^2}{4^3 - 3^3} \,.$$

Hence, the average numbers of costumers in the shop (in the long run) is:

$$\bar{n} = 0\bar{\pi}_0 + 1\bar{\pi}_1 + 2\bar{\pi}_2 = \frac{3 \cdot 4}{4^3 - 3^3} + 2\frac{3^2}{4^3 - 3^3} = \frac{30}{37} \,.$$

(b) The (average) proportion of potential costumers entering the shop is equal to the fraction of time that the shop has 0 or 1 costumers, which is the same as

$$\bar{\pi}_0 + \bar{\pi}_1 = \frac{4^2}{4^3 - 3^3} + \frac{3 \cdot 4}{4^3 - 3^3} = \frac{28}{37} \simeq 0.76 \,.$$

(c) If the barber works at double speed, the new equilibrium distribution is

$$\bar{\pi}'_0 = 5\frac{8^2}{8^3 - 3^3} \ ,\ \ \bar{\pi}'_1 = 5\frac{3 \cdot 8}{8^3 - 3^3} \ ,\ \ \bar{\pi}'_2 = 5\frac{3^2}{8^3 - 3^3} \,.$$

The (average) proportion of potential costumers entering the shop is equal to

$$\bar{\pi}'_0 + \bar{\pi}'_1 = 5\frac{8^2}{4^3 - 3^3} + 5\frac{3 \cdot 8}{8^3 - 3^3} = \frac{440}{485} \simeq 0.91 \,.$$

Hence, his business has a growth of $0.91/0.76 \simeq 1.19$, approximately of %20.

Problem 3: Let M_n, $n \geq 0$, with $M_0 = 0$, be a martingale, and let $X_n = M_n - M_{n-1}$, $n \geq 1$. Prove that $Var(M_n) = \sum_{i=0}^{n} Var(X_i)$.

Solution: First, we note that $\mathbb{E}[M_n] = 0$, and $\mathcal{E}[X_n] = \mathbb{E}[M_n - M_{n-1}] = 0$, by definition of Martingale. Moreover, for $m < n$, we have

$$\begin{aligned} &\mathbb{E}[X_m X_n] = \mathbb{E}[(M_m - M_{m-1})(M_n - M_{n-1})] = \mathbb{E}[\mathbb{E}[(M_m - M_{m-1})(M_n - M_{n-1})|\mathcal{F}_m]] \\ &= \mathbb{E}[(M_m - M_{m-1})\mathbb{E}[(M_n - M_{n-1})|\mathcal{F}_m]] = \mathbb{E}[(M_m - M_{m-1})(M_m - M_m)] = 0 \,. \end{aligned}$$

We can write M_n as a telescopic sum, to get

$$M_n = M_n - M_0 = (M_n - M_{n-1}) + (M_{n-1} - M_{n-2}) + \cdots + (M_1 - M_0) = X_1 + \cdots + X_n \,.$$

Hence,

$$Var(M_n) = \mathbb{E}[M_n^2] = \mathbb{E}[(\sum_{i=1}^{n} X_i)^2] = \sum_{i=1}^{n} \mathbb{E}X_i^2] + 2\sum_{i<j=1}^{n} \mathbb{E}[X_i X_j] = \sum_{i=1}^{n} Var(X_i) \,.$$

Problem 4: Let $X_1, X_2, X_3, \ldots$ be independent identically distributed random variables. Let $m(t) = \mathbb{E}[e^{tX_1}] < \infty$ be the moment generating function of X_1. Show that

$$M_n = m(t)^{-n} e^{t(X_1 + \cdots + X_n)} \,,\ n \geq 0 \,,$$

is a martingale.

Solution: If $\mathcal{F}_n$ denotes the "information contained in the r.v.'s $X_1, \ldots, X_n$", clearly M_n is $\mathcal{F}_n$-measurable. We want to show that $\mathbb{E}[M_{n+1}|\mathcal{F}_n] = M_n$. We have

$$\begin{aligned} &\mathbb{E}[M_{n+1}|\mathcal{F}_n] = \mathbb{E}[m(t)^{-n-1} e^{t(X_1 + \cdots + X_n + X_{n+1})}|\mathcal{F}_n] = m(t)^{-n-1}\mathbb{E}[e^{t(X_1 + \cdots + X_n)} e^{tX_{n+1}}|\mathcal{F}_n] \\ &= m(t)^{-n-1} e^{t(X_1 + \cdots + X_n)}\mathbb{E}[e^{tX_{n+1}}|\mathcal{F}_n] = m(t)^{-n-1} e^{t(X_1 + \cdots + X_n)}\mathbb{E}[e^{tX_{n+1}}] \\ &= m(t)^{-n-1} e^{t(X_1 + \cdots + X_n)}\mathbb{E}[e^{tX_1}] = m(t)^{-n-1} e^{t(X_1 + \cdots + X_n)} m(t) = m(t)^{-n} e^{t(X_1 + \cdots + X_n)} = M_n \,. \end{aligned}$$

Problem 5: Let B_t, ≥ 0 be the standard Brownian motion. Find the probability density function of the following random variables:

(1) $|B_t|$,
(2) $|\max_{0\leq s\leq t} B_s|$
(3) $\max_{0\leq s\leq t} B_s - B_t$.

Solution: (a) B_t is a normal random variable of variance t. Hence

$$\mathbb{P}[|B_t| < x] = \mathbb{P}[-x < B_t < x] = 2\mathbb{P}[0 \leq B_t < x] = 2\int_0^x \frac{1}{\sqrt{2\pi t}} e^{-y^2/2t} dy\,,$$

and the corresponding density function is

$$f_{|B_t|}(x) = \frac{d}{dx}\mathbb{P}[|B_t| < x] = \sqrt{\frac{2}{\pi t}} e^{-x^2/2t}\,.$$

(b) Since $B_0 = 0$, we clearly have $\max_{0\leq s\leq t} B_s \geq 0$. By the reflection principle, for $x > 0$

$$\mathbb{P}[\max_{0\leq s\leq t} B_s \geq x] = 2\mathbb{P}[B_t \geq x]\,.$$

Hence,

$$\begin{aligned} &\mathbb{P}[\max\nolimits_{0\leq s\leq t} B_s < x] = 1 - \mathbb{P}[\max\nolimits_{0\leq s\leq t} B_s \geq x] = 1 - 2\mathbb{P}[B_t \geq x] \\ &= 1 - 2(1 - \mathbb{P}[B_t < x]) = 2\mathbb{P}[B_t < x] - 1\,, \end{aligned}$$

and

$$\begin{aligned} &f_{|\max_{0\leq s\leq t} B_s|}(x) = \tfrac{d}{dx}\mathbb{P}[\max\nolimits_{0\leq s\leq t} B_s < x] = \tfrac{d}{dx}\big(2\mathbb{P}[B_t < x] - 1\big) \\ &= 2\tfrac{d}{dx}\tfrac{1}{\sqrt{2\pi t}}\textstyle\int_{-\infty}^x e^{-y^2/2t}dy = \sqrt{\tfrac{2}{\pi t}} e^{-x^2/2t}\,. \end{aligned}$$

(c) We have, for $x \geq 0$,

$$\begin{aligned} &\mathbb{P}[\max\nolimits_{0\leq s\leq t} B_s \geq B_t + x] = \textstyle\int_{-\infty}^{\infty} \mathbb{P}[\max\nolimits_{0\leq s\leq t} B_s \geq a + x \,|\, B_t = a] f_{B_t}(a) da \\ &= \textstyle\int_{-\infty}^{-x} f_{B_t}(a) da + \int_{-x}^{\infty} \mathbb{P}[\max\nolimits_{0\leq s\leq t} B_s \geq a + x \,|\, B_t = a] f_{B_t}(a) da\,. \end{aligned}$$

With an argument similar to the one use to prove the reflection principle, it is not hard to see that

$$\begin{aligned} &\mathbb{P}[\max\nolimits_{0\leq s\leq t} B_s \geq a + x \,|\, B_t = a] f_{B_t}(a) da = \mathbb{P}[\max\nolimits_{0\leq s\leq t} B_s \geq a + x,\, B_t \in (a, a + da)] \\ &= \mathbb{P}[B_t \in (a, a + da) \,|\, \max\nolimits_{0\leq s\leq t} B_s \geq a + x]\mathbb{P}[\max\nolimits_{0\leq s\leq t} B_s \geq a + x] \\ &= \mathbb{P}[B_t \in (a, a + da) \,|\, \tau_{a+x} \leq t]\mathbb{P}[\max\nolimits_{0\leq s\leq t} B_s \geq a + x] \\ &= \mathbb{P}[B_t \in (a + 2x, a + 2x + da) \,|\, \tau_{a+x} \leq t]\mathbb{P}[\max\nolimits_{0\leq s\leq t} B_s \geq a + x] \\ &= \mathbb{P}[\max\nolimits_{0\leq s\leq t} B_s \geq a + x,\, B_t \in (a + 2x, a + 2x + da)] = f_{B_t}(a + 2x) da\,. \end{aligned}$$

Hence,

$$\begin{aligned} &\mathbb{P}[\max\nolimits_{0\leq s\leq t} B_s \geq B_t + x] = \textstyle\int_{-\infty}^{-x} f_{B_t}(a) da + \int_{-x}^{\infty} f_{B_t}(a + 2x) da \\ &= \textstyle\int_{-\infty}^{-x} f_{B_t}(a) da + \int_{x}^{\infty} f_{B_t}(b) db = 1 - \int_{-x}^{x} f_{B_t}(a) da = 1 - 2\int_0^x f_{B_t}(a) da\,. \end{aligned}$$

Hence,

$$f_{\max_{0\leq s\leq t} B_s - B_t}(x) = \tfrac{d}{dx}(1 - \mathbb{P}[\max\nolimits_{0\leq s\leq t} B_s - B_t \geq x]) = \sqrt{\tfrac{2}{\pi t}} e^{-x^2/2t}\,.$$

In conclusion, all three variables in (a), (b) and (c) have the same probability density.

Problem 6: . Let $X_t = e^{at + bB_t}$, where B_t be the standard Brownian motion, and $a, b \geq 0$ are constants.

(1) Find the probability density function of X_t.
(2) Compute dX_t.
(3) For which values of a, b is X_t a martingale?

Solution: (a) We have

$$\mathbb{P}[X_t \leq x] = \mathbb{P}[e^{at + bB_t \leq x}] = \mathbb{P}[at + bB_t \leq \log x] = \mathbb{P}[B_t \leq \frac{\log x - at}{b}] = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\frac{\log x - at}{b}} e^{-y^2/2t} dy\,.$$

Computing the derivative, we thus get

$$f_{X_t} = \frac{d}{dx}\mathbb{P}[X_t \leq x] = \frac{1}{\sqrt{2\pi t}} e^{-(\log x - at)^2/2b^2 t} \frac{1}{bx}\,.$$

(b) By Ito formula, we have

$$dX_t = \frac{\partial}{\partial B_t} X_t dB_t + (\frac{d}{dt} X_t + \frac{1}{2} \frac{\partial^2}{\partial B_t^2} X_t) dt = b X_t dB_t + (a + \frac{1}{2} b^2) X_t dt \,.$$

(c) X_t is a Martingale when the dt term disappears, i.e. when $a = -\frac{1}{2} b^2$.